Spectral functions for some product of selfajoint operators

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Let $\mathcal{H}$ be a Hilbert space with a scalar product.

Linear operator $G : \mathcal{H} \to \mathcal{H}$ is selfajoint and continuous. Let number 0 is not an eigenvalue of the operator G, ($0 \notin \sigma_p(G)$).

We consider a form $[x, y] := (Gx, y)$ $x, y \in \mathcal{H}$.

Hilbert space $\mathcal{H}$ with the form $[x, y]$ is a singular G-space, if 0 is a point of continuous spectrum of operator G.

Hilbert space $\mathcal{H}$ with the form $[x, y]$ is a regular G-space, if 0 is a regular point of operator G.

Definition. 1. *Linear continuous operator* $A : \mathcal{H} \to \mathcal{H}$ *is* G*-selfajoint, if* $[Ax, y] = [x, Ay]$ *for any* $x, y \in \mathcal{H}$.

Let ring $\mathfrak{R}$ is generated by some intervals real axis, ring $\mathfrak{R}$ including the interval $(-\infty, +\infty)$.

Let E is the homomorphism, mapping the ring $\mathfrak{R}$ to set of G-selfajoint projectors $\{E(\triangle)\}$. Any projectors from this set $\{E(\triangle)\}$ satisfy the following conditions :

- $E(\varnothing) = 0$, $E((-\infty, +\infty)) = I$.
- $E(\triangle \cap \triangle^{'}) = E(\triangle)E(\triangle^{'})$.
- $E(\triangle \cup \triangle^{'}) = E(\triangle) + E(\triangle^{'})$ if $\triangle \cap \triangle^{'} = \varnothing$, where $\triangle, \triangle^{'} \in \mathfrak{R}$.

Denote $\sigma(E)$ the support of this homomorphism.

Denote $\mathfrak{P}^{+}$ ($\mathfrak{P}^{-}$) the set of linear manifolds L, which satisfy the condition: for any $x \in L$: $[x, x] \geq 0$ ($[x, x] \leq 0$).

$\lambda \in \sigma(E)$ is the point of positive (negative) type, if there is an interval $\triangle \in \mathfrak{R}$, which contain λ, and $E(\triangle)\mathcal{H} \in \mathfrak{P}^+$ $(E(\triangle)\mathcal{H} \in \mathfrak{P}^-)$.

Symbols $\sigma_+(E)$ and $\sigma_-(E)$ denote the set of points of positive and negative types.

Let $s = \{a_j\}_{j=1}^n$ be finite set of real points. Ring $\mathfrak{R}(s)$ generated by intervals, for which points from s are not bounds points.

Definition. 2. *Homomorphism E, defined on $\mathfrak{R}(s)$, is G-spectral function with the set $s(E)$ of the critical points, if from the conditions $\lambda_0, \mu_0 \notin s(E),\ \ \mu_0 < \lambda_0$ and $\mu \downarrow \mu_0$, follows that the sequence operators $\{E((\mu, \lambda_0])\}$ converge in the strong operators topology and the limit coincide with $E((\mu_0, \lambda_0])$:*

$$s - \lim_{\mu \downarrow \mu_0} (E((\mu, \lambda_0])) = E((\mu_0, \lambda_0])$$

Definition. 3. *Point $a \in s(E)$ is the regular critical point of the G-spectral function, if the limits:*

- $s - \lim_{\lambda \uparrow a} E((-\infty, \lambda])$
- $s - \lim_{\lambda \downarrow a} E((\lambda, +\infty))$

exist in the strong operators topology, otherwise point a is the singular critical point.

If set $s(E)$ does not contain the singular critical points, then E is regular G-spectral function.

Definition. 4. *G-spectral function E is the eigen G-spectral function of the G-selfajoint operators A, if:*

- $AE(\triangle) = E(\triangle)A$

- $\sigma(A|E(\triangle)\mathcal{H}) \subset \overline{\triangle}$

- *if* $\triangle \in \mathfrak{R}$ *and* $\triangle \cap s(E) = \varnothing$ *then:* $AE(\triangle) = \int\limits_{\triangle} t dE_t$

If the set $\triangle$ is bounded, then the integral converg in norm, , if the set $\triangle$ is unbounded, then the integral converge in the strong operators topology.

Let $s(A)$ be the set of the critical points of eigen G-spectral functions of operator A.

It is easy to check that, if the linear operator $A : \mathcal{H} \to \mathcal{H}$ is continuous and selfajoint, then the linear operator AG is G-selfajoint in the G-space $\mathcal{H}$.

The existence of eigen G-spectral function of the continuous G-selfajoint operators T in G-space for case of regular G-space (when $0 \notin \sigma(G)$) was shown by H.Langer. Therefore, we study the case of the singular G-space (when $0 \in \sigma_c(G)$).

In this work, in case of singular G-space, we show existence of eigen G-spectral function of continuous G-selfajoint operators T, where $T = AG$. The linear continuous operator $A : \mathcal{H} \to \mathcal{H}$ is selfajoint and nonnegative: $A \geq 0$.

The main result of our work is the theorem 1 and 2. In the proof of the theorem 1 we follow J.Bognar.

Theorem. 1. *Let the linear operators* $A : \mathcal{H} \to \mathcal{H}$, $G : \mathcal{H} \to \mathcal{H}$ *are continuous and selfajoint. If the operator* A *is nonnegative and* $0 \notin \sigma_p(G)$, *then for any real number* $\lambda \neq 0$ *there is the unique* G*-selfajoint projector* E_λ, *and the function* $\lambda \to E_\lambda$ *satisfy the following conditions:*

1. *If* $\lambda \leq \mu$, *then* $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$.

2. *If* $\lambda < \mu < 0$, *then* $[E_\lambda x, x] \geq [E_\mu x, x]$, *and if* $\mu > \lambda > 0$, *then* $[E_\mu x, x] \geq [E_\lambda x, x]$ *for any* $x \in \mathcal{H}$.

3. *If* $\lambda < -\|A\|\|G\|$, *then* $E_\lambda = 0$, *and if* $\lambda > \|A\|\|G\|$, *and* $E_\lambda = I$.

4. *If* $\lambda \neq 0$*, then exist the limit in the strongly operator topology* $s\text{-}\lim_{\mu \downarrow \lambda} E_\mu = E_{\lambda+0}$ *and his coincide with* E_λ*.*

5. *If* T *is the linear bounded operator and* T *commutating with the operator* AG*, then* $TE_\lambda = E_\lambda T$*.*

6. $\sigma(AG|E_\lambda\mathcal{H}) \subset (-\infty, \lambda]$*, and* $\sigma(AG|(I - E_\lambda)\mathcal{H}) \subset [\lambda, +\infty)$*.*

An integral $\int\limits_{-\|A\|\|G\|}^{\|A\|\|G\|} \nu dE_\nu$ *is the convergent integral in the strongly operator topology with the singular point* $\lambda = 0$*.*

Consider the polar representation of the operator G, $G = |G|J$. If G is the selfajoint operator and $0 \in \sigma_c(G)$, then J is the unitary and selfajoint operator.

Introduce the operator $B := JAJ$. The operator A is nonnegative, then operator $B \geq 0$, therefore there exist the operator $B^{1/2}$.

Introduce the selfajoint operator C such that

$$C := B^{1/2} G B^{1/2}$$

Let $\{F_\lambda\}_{-\infty}^{+\infty}$ be the right continuous spectral function for the operator C.

Introduce the operator C_λ:

$$C_\lambda = \begin{cases} C|F_\lambda, & \text{if } \lambda < 0; \\ C|(I - F_\lambda), & \text{if} \lambda > 0. \end{cases}$$

Introduce the function E_λ such that:

$$E_\lambda = \begin{cases} J B^{1/2} C_\lambda^{-1} F_\lambda B^{1/2} |G|, & \text{if } \lambda < 0; \\ I - J B^{1/2} C_\lambda^{-1} (I - F_\lambda) B^{1/2} |G|, & \text{if } \lambda > 0. \end{cases}$$

Theorem. 2. *If $A : \mathcal{H} \to \mathcal{H}$ and $G : \mathcal{H} \to \mathcal{H}$ is continuous selfajoint operators, A is the nonnegative and $0 \notin \sigma_p(G)$. Then the operator $AG : \mathcal{H} \to \mathcal{H}$ have the eigen G-spectral functions E with critical point $\lambda = 0$. And the next statements is hold:*

- $\sigma(E) \backslash \{0\} = \sigma_+(E) \cup \sigma_-(E)$
- $\sigma_+(E) = (0, +\infty) \cap \sigma(E)$
- $\sigma_-(E) = (-\infty, 0) \cap \sigma(E)$

If $\triangle = (\alpha, \beta]$ (or $[\alpha, \beta), (\alpha, \beta), [\alpha, \beta]$) and $\alpha \neq 0$, $\beta \neq 0$, then $E(\triangle) := E_\beta - E_\alpha$. The mapping $\triangle \to E(\triangle)$ is define the proper G-spectral functions E with critical point $\lambda = 0$ for the operator AG.

In the general case, when $T : \mathcal{H} \to \mathcal{H}$ is continuous G-selfajoint and G-nonnegative operator, T does not have the representation $T = AG$, where A and G is satisfy the condition of theorem 1, operator T may not have G-spectral function.

Further we are show an example of the continuous G-selfajoint and G-nonnegative operator T, which has no G-spectral function.

Example: Let $T : Dom(T) \to \mathcal{H}$ be hermitian, uniformly positive operator, $\overline{Dom(T)} = \mathcal{H}$, and his defects indexes equal $(1, 1)$.

Introduce the operator $G := |T|^{-1}$. The operator G continuous and selfajoint, and at the same time $0 \in \sigma_c(|T|^{-1}) = \sigma_c(G)$.

As $Ran(|T|^{-1}) = Dom(T) \neq \mathcal{H}$ and $\overline{Dom(T)} =$ $= \mathcal{H}$, therefore $0 \in \sigma_c(|T|^{-1})$.

If the defects indexes of the operator T coincide, then the operator T can be extended to selfajoint, uniformly positive operator $\widetilde{T}$. Denote $A := \widetilde{T}$. The operator AG is the continuous G-selfajoint and G-nonnegative.

As $AG = \widetilde{T}|T|^{-1}$, $\widetilde{T}|DomT = T$ and $Ran(|T|^{-1}) = = Dom(T) \subset Dom(\widetilde{T})$, we have:

$$AG = \widetilde{T}|T|^{-1} = T|T|^{-1}$$

Consider the polar representation of operator T: $T = V|T|$ whence $V = T|T|^{-1} = AG$. If T is hermitian operator, then V — semi-unitary operator, therefore:

$$\sigma(AG) = \sigma(V) = \{\lambda : \lambda \in \mathbb{C}, |\lambda| \leq 1\}$$

Assuming the contrary: the operator AG have the proper G-spectral function E. Hence $\sigma(AG) = = \sigma(AG|E((-\infty, +\infty))\mathcal{H}) \subset \mathbb{R}$, as $\sigma(AG) = = \{\lambda : \lambda \in \mathbb{C}, |\lambda| \leq 1\}$, we have contradiction.

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