Coupling method in the theory of generalized resolvents of symmetric operators

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joint work with
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[1] V. Derkach, S. Hassi, M. Malamud and H. de Snoo, Boundary relations and their Weyl families, Trans.Amer.Math.Soc. 358 (2006), 5351-5400.

Boundary triplets and Weyl functions

Let A be a closed symmetric operator in a Hilbert space $\mathfrak{H}$. Let A^* be the adjoint linear relation

$$\{g, g'\} \in A^* \Longleftrightarrow (Af, g) - (f, g') = 0 \; \forall f \in \operatorname{dom} A$$

Definition 1 *(A.Kochubej '75, M. Malamud '92) A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H}$ is a Hilbert space and $\Gamma_i : A^* \to \mathcal{H}$, $i = 0, 1$, is said to be a boundary triplet for A^*, if for all $\widehat{f} = \{f, f'\}, \widehat{g} = \{g, g'\} \in A^*$;*

$$(f', g)_{\mathfrak{H}} - (f, g')_{\mathfrak{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}} \qquad (1)$$

and $\Gamma := \{\Gamma_0, \Gamma_1\} : A^* \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

Define a selfadjoint extension A_0 of A by

$$A_0 = \ker \Gamma_0.$$

Definition 2 *(DM '85) The abstract Weyl function of A corresponding to the boundary triplet Π*

$$\Gamma_1 f_\lambda = M(\lambda)\Gamma_0 f_\lambda, \quad f_\lambda \in \mathfrak{N}_\lambda := \ker(A^* - \lambda), \qquad (2)$$

the $\gamma -$ field

$$\gamma(\lambda) := (\Gamma_0|_{\mathfrak{N}_\lambda})^{-1} : \mathbb{C} \setminus \mathbb{R} \to \mathfrak{N}_\lambda.$$

Nevanlinna functions

It follows from (1) that $M(\lambda)$ satisfies

$$M(\lambda) - M(\mu)^* = (\lambda - \bar{\mu})\gamma(\mu)^*\gamma(\lambda), \quad \lambda, \mu \in \rho(A_0).$$

This implies that $M(\cdot) \in R[\mathcal{H}]$:
1) $M(\cdot) : \mathbb{C} \setminus \mathbb{R} \to [\mathcal{H}]$ is holomorphic;
2) $M(\lambda)^* = M(\bar{\lambda})$;
3) $\text{Im } M(\lambda)\text{Im }(\lambda) \geq 0$.
Subclasses of $R[\mathcal{H}]$: $R^u[\mathcal{H}] \subset R^s[\mathcal{H}] \subset R[\mathcal{H}]$

$$M(\cdot) \in R^s[\mathcal{H}] \Longleftrightarrow 0 \notin \sigma_p(\text{Im } M(\lambda)) \text{ for all } \lambda \in \mathbb{C} \setminus \mathbb{R};$$

$$M(\cdot) \in R^u[\mathcal{H}] \Longleftrightarrow 0 \in \rho(\text{Im } M(\lambda)) \text{ for all } \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

It is known (LT '77, DM '95), that every uniformly strict Nevanlinna function is the Weyl function of a symmetric operator A, corresponding to a boundary triplet Π.

Nevanlinna families

A family $\tau(\lambda)$ is called a Nevanlinna family and is written as $\tau \in \widetilde{R}(\mathcal{H})$, if:
1) $\tau(\lambda)$ is a maximal dissipative linear relation for all $\lambda \in \mathbb{C}_+$;
2) $\tau(\lambda)^* = \tau(\bar{\lambda})$, $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$;
3) $(\tau(\lambda) + i)^{-1}$ is holomorphic on $\mathbb{C}_+$.

Classical Weyl-Titchmarsh function

Let the Sturm-Liouville operator $\ell = -D^2 + q$ on $(0, \infty)$ be in the limit-point case at ∞. If A is a minimal operator generated by ℓ in $L_2(0, \infty)$, then def $A = \dim \mathfrak{N}_\lambda(A) = 1$. The boundary triplet is given by

$$\Gamma_0 f = f(0),\ \Gamma_1 f = f'(0)\ f \in \operatorname{dom} A^*$$

Let $u(x, \lambda)$, $v(x, \lambda)$ be solutions of $\ell(f) = \lambda f$ such that

$$u(0, \lambda) = 1, \quad u'(0, \lambda) = 0;$$
$$v(0, \lambda) = 0, \quad v'(0, \lambda) = 1.$$

By Weyl theorem $\exists!$ $m(\lambda)$ such that

$$(f_\lambda =) u(x, \lambda) + m(\lambda) v(x, \lambda) \in L_2(0, \infty).$$

Since $\Gamma_0 f_\lambda = 1$, $\Gamma_1 f_\lambda = m(\lambda)$ one obtains

$$M(\lambda) = m(\lambda).$$

Canonical selfadjoint extensions of A: $\widetilde{A} = A^*|_{\operatorname{dom} \widetilde{A}}$

$$\operatorname{dom} \widetilde{A} = \{f \in \operatorname{dom} A^* : y'(0) = \theta y(0)\}, \quad \theta \in \mathbb{R} \cup \{\infty\}.$$

Canonical and generalized resolvents of A

If $\dim \mathcal{H} = 1$ s.a. extensions of A are parametrized by

$$\widetilde{A}_\theta = \ker(\Gamma_1 + \theta\Gamma_0)\ (\theta \in \mathbb{R}),\ \widetilde{A}_\infty = A_0 = \ker\Gamma_0.$$

Canonical resolvents of A are parametrized by

$$(\widetilde{A}_\theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)(\theta + M(\lambda))^{-1}\gamma(\bar{\lambda})^*. \quad (3)$$

Generalized resolvent of A is a compressed resolvent

$$\mathbf{R}_\lambda = P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1}|_{\mathfrak{H}},$$

of a selfadjoint extension $\widetilde{A}$ of A in $\widetilde{\mathfrak{H}}(\supset \mathfrak{H})$,
$P_{\mathfrak{H}}$ is the orthogonal projection onto $\mathfrak{H}$ in $\widetilde{\mathfrak{H}}$.
Description of generalized resolvents (M.G. Kreĭn '44)

$$\mathbf{R}_\lambda = (A_0 - \lambda)^{-1} - \gamma(\lambda)(\tau(\lambda) + M(\lambda))^{-1}\gamma(\bar{\lambda})^*, \quad (4)$$

where $\tau(\in R)$ is a Nevanlinna function.

Proof of Kreĭn-Najmark formula. Let $\tau \in R$. Let us construct a s.a. extension $\widetilde{A}$ of A such that (4) holds.

There are a symmetric operator S_2 in $\mathfrak{H}''$ and a boundary triplet $\{\mathbb{C}, \Gamma_0'', \Gamma_1''\}$ for S_2^* such that the corresponding Weyl function is τ. Let $H(\supset S := A \oplus S_2)$ be a symmetric operator in $\mathfrak{H} \oplus \mathfrak{H}''$ with the adjoint

$$H^* = \left\{ \widehat{f}_1 \oplus \widehat{f}_2 \in A^* \oplus S_2^* : \Gamma_0 \widehat{f}_1 - \Gamma_0'' \widehat{f}_2 = 0 \right\}.$$

Then $\tau(\lambda) + M(\lambda)$ is a Weyl function for H^*, corresponding to the BT $\{\mathbb{C}, \Gamma_0, \Gamma_1 \oplus \Gamma_1''\}$. Then $\widetilde{A} = \ker(\Gamma_1 \oplus \Gamma_1'')$ is a selfadjoint extension of H

$$\widetilde{A} = \left\{ \widehat{f}_1 \oplus \widehat{f}_2 \in S^* : \Gamma_0 \widehat{f}_1 - \Gamma_0'' \widehat{f}_2 = \Gamma_1 \widehat{f}_1 + \Gamma_1'' \widehat{f}_2 = 0 \right\}$$

due to (3) its resolvent is given by

$$\begin{aligned} (\widetilde{A} - \lambda)^{-1} &= \mathrm{diag}((\widetilde{A}_\infty - \lambda)^{-1}, (\widetilde{A}_\infty'' - \lambda)^{-1}) \\ &- \begin{pmatrix} \gamma(\lambda) \\ \gamma''(\lambda) \end{pmatrix} (\theta + M(\lambda))^{-1} \begin{pmatrix} \gamma(\bar{\lambda})^* & \gamma''(\bar{\lambda})^* \end{pmatrix}. \end{aligned} \tag{5}$$

Compression of(5) to $\mathfrak{H}$ gives (4).

Problem Prove the Kreĭn-Najmark formula via coupling method for infinite indices.

Difficulties If $\tau \in \widetilde{R}(\mathcal{H}) \setminus R^u[\mathcal{H}]$, then either $\tau(\lambda)$ is multivalued, or $\tau(\lambda)$ is unbounded, or $\Im \tau(\lambda)$ is not invertible.

Tools Generalize the notion of boundary triplet in order that arbitrary Nevanlinna family to be realized as the corresponding Weyl family.

Unitary relations in Kreĭn spaces

Let $J_{\mathfrak{H}}$, $J_{\mathcal{H}}$ be signature operators in $\mathfrak{H}^2$ and $\mathcal{H}^2$:
$J_{\mathfrak{H}} := \begin{pmatrix} 0 & -iI_{\mathfrak{H}} \\ iI_{\mathfrak{H}} & 0 \end{pmatrix}$, $J_{\mathcal{H}} := \begin{pmatrix} 0 & -iI_{\mathcal{H}} \\ iI_{\mathcal{H}} & 0 \end{pmatrix}$.
$\Gamma : (\mathfrak{H}^2, J_{\mathfrak{H}}) \to (\mathcal{H}^2, J_{\mathcal{H}})$ is called a linear relation, iff Γ is a linear subspace of $\mathfrak{H}^2 \times \mathcal{H}^2$. A linear relation Γ is called *isometric*, if:

$$(J_{\mathcal{H}} \Gamma f, \Gamma g) = (J_{\mathfrak{H}} f, g) \ \forall f, g \in \operatorname{dom} \Gamma; \qquad (6)$$

or, equivalently, $\Gamma^{-1} \subset \Gamma^{[*]} := J_{\mathfrak{H}} \Gamma^* J_{\mathcal{H}}$.

Definition 3 *(Shmul'jan, '76) A linear relation* $\Gamma : (\mathfrak{H}^2, J_{\mathfrak{H}}) \to (\mathcal{H}^2, J_{\mathcal{H}})$ *is called unitary, if* $\Gamma^{-1} = \Gamma^{[*]}$.

Proposition 4 *Let* Γ *be a unitary relation from the Kreĭn space* $(\mathfrak{H}, j_{\mathfrak{H}})$ *to the Kreĭn space* $(\mathcal{H}, j_{\mathcal{H}})$*. Then* $\operatorname{dom} \Gamma$ *is closed if and only if* $\operatorname{ran} \Gamma$ *is closed. Moreover,*

$$\ker \Gamma = (\operatorname{dom} \Gamma)^{[\perp]}, \quad \operatorname{mul} \Gamma = (\operatorname{ran} \Gamma)^{[\perp]}.$$

An isometric operator from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$ is called a *standard* unitary operator, if $\operatorname{dom} \Gamma = \mathfrak{H}^2$, $\operatorname{ran} \Gamma = \mathcal{H}^2$. Clearly, Γ is a unitary relation if at least one of this conditions holds.

Example 5 *If* $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is a boundary triplet, then* $\Gamma : (\mathfrak{H}^2, J_{\mathfrak{H}}) \to (\mathcal{H}^2, J_{\mathcal{H}})$ *is a unitary relation, since* (1) *is equivalent to* (6) *and* $\operatorname{ran} \Gamma = \mathcal{H}^2$.

Boundary relations and Weyl families

Definition 6 *(DHMS'06) A linear relation $\Gamma : (\mathfrak{H}^2, J_{\mathfrak{H}}) \mapsto (\mathcal{H}^2, J_{\mathcal{H}})$ is called a boundary relation for S^*, if:*
1) Γ is a unitary relation; 2) $S = \ker \Gamma$.

In general $T := \operatorname{dom} \Gamma \neq S^*$. Let $\mathfrak{N}_\lambda(T) = \ker(T - \lambda)$ and $\widehat{\mathfrak{N}}_\lambda(T) = \{ \{f, \lambda f\} \in T : f \in \mathfrak{N}_\lambda(T) \}$.

Definition 7 *Weyl family $\tau(\lambda)$ of S corresponding to the boundary relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is defined by*

$$\tau(\lambda) := \Gamma(\widehat{\mathfrak{N}}_\lambda(T)), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{7}$$

Boundary relation Γ is called *minimal*, if

$$\mathfrak{H} = \mathfrak{H}_{min} := \overline{\operatorname{span}}\{ \mathfrak{N}_\lambda(T) : \lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \}.$$

Theorem 8 *(DHMS'06) Let $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation for S^*. Then the corresponding Weyl family $M(\cdot)$ belongs to the class $\widetilde{R}(\mathcal{H})$.*

Conversely, if $M(\cdot)$ belongs to the class $\widetilde{R}(\mathcal{H})$ then there exists a unique (up to unitary equivalence) minimal boundary relation whose Weyl function coincides with $M(\cdot)$.

Induced Boundary Relation

Let $\widetilde{A}$ be a selfadjoint linear relation in the orthogonal sum $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and let

$$S_j = \widetilde{A} \cap \mathfrak{H}_j^2, \quad T_j = \left\{ \begin{pmatrix} P_j\varphi \\ P_j\varphi' \end{pmatrix} : \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix} \in \widetilde{A} \right\}. \tag{8}$$

$\widetilde{A}$ is called a *minimal* selfadjoint extension of S_1, if

$$\widetilde{\mathfrak{H}} = \overline{\mathrm{span}} \left\{ \mathfrak{H}_1 + (\widetilde{A} - \lambda)^{-1}\mathfrak{H}_1 : \lambda \in \rho(\widetilde{A}) \right\}. \tag{9}$$

Theorem 9 *1) Let A be a symmetric operator in $\mathfrak{H}_1$, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary BT for A^*. If $\widetilde{A} = \widetilde{A}^*$ is a minimal selfadjoint exit space extension of A in $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and S_2, T_2 are defined by (8), then the linear relation $\Gamma'' : \mathfrak{H}_2^2 \to \mathcal{H}^2$ defined by*

$$\Gamma'' = \left\{ \left\{ \widehat{f}_2, \begin{pmatrix} \Gamma_0 \widehat{f}_1 \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix} \right\} : \widehat{f}_1 \oplus \widehat{f}_2 \in \widetilde{A},\ \widehat{f}_1 \in T_1,\ \widehat{f}_2 \in T_2 \right\} \tag{10}$$

is a minimal boundary relation for S_2^*.
2) Conversely, if S_2 is a simple symmetric operator in $\mathfrak{H}_2$ and $\Gamma'' : \mathfrak{H}_2^2 \to \mathcal{H}^2$ is a minimal boundary relation for S_2^*, then the linear relation $\widetilde{A}$ defined by

$$\widetilde{A} = \left\{ \widehat{f}_1 \oplus \widehat{f}_2 \in A^* \oplus S_2^* : \left\{ \widehat{f}_2, \begin{pmatrix} \Gamma_0 \widehat{f}_1 \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix} \right\} \in \Gamma'' \right\} \tag{11}$$

is a minimal selfadjoint extension of A which satisfies $\widetilde{A} \cap \mathfrak{H}_2^2 = S_2$. Moreover, the compressed resolvent of $\widetilde{A}$ is calculated by the Kreĭn-Najmark formula.

Najmark extensions

Definition 10 *(Najmark) Let A be a densely defined symmetric operator in $\mathfrak{H}$ and let $\widetilde{A}$ be a minimal exit space extension of A acting in $\widetilde{\mathfrak{H}}$ ($\supset \mathfrak{H}$). Then*
(i) $\widetilde{A} \in Naj_1(A)$ *if* $\mathrm{dom}\, \widetilde{A} \cap \mathfrak{H} = \mathrm{dom}\, \widetilde{A}$;
(ii) $\widetilde{A} \in Naj_2(A)$ *if* $\mathrm{dom}\, \widetilde{A} \cap \mathfrak{H} = \mathrm{dom}\, A$;
(iii) $\widetilde{A} \in Naj_2(A)$ *if* $\mathrm{dom}\, A \subsetneqq \mathrm{dom}\, \widetilde{A} \cap \mathfrak{H} \subsetneqq \mathrm{dom}\, \widetilde{A}$.

Note that a first type extension $\widetilde{A}$ is just a canonical extension of A, acting in $\widetilde{\mathfrak{H}} = \mathfrak{H}$.

Theorem 11 $\widetilde{A} \in Naj_2(A) \Leftrightarrow S_1 = A \& \mathrm{mul}\, T_2 = \{0\}$

Proposition 12 *Let* $n_\pm(A) = n < \infty$.
Then $\widetilde{A} \in Naj_2(A)$ *iff* $\tau(\cdot) \in R^u[\mathcal{H}]$ *and*

$$\lim_{y \downarrow \infty} y^{-1} \tau(iy) = 0, \tag{12}$$

$$\lim_{y \downarrow \infty} y \cdot \mathrm{Im}\, (\tau(iy)h, h) = \infty, \quad h \in \mathcal{H} \setminus \{0\}. \tag{13}$$

Theorem 13 *Let* $n_\pm(A) = \infty$. *Then* $\widetilde{A} \in Naj_2(A)$ *iff* $\tau(\cdot) \in R^s(\mathcal{H})$ *and the function*

$$\tau^{(1)}(\lambda) = -(\tau(\lambda) - 1/\lambda)^{-1}$$

satisfies the limit conditions in (12), (13).

Najmark extensions of nondensely defined operator

Definition 14 *Let A be a nondensely defined symmetric operator in $\mathfrak{H}$ and let $\widetilde{A}$ be a minimal exit space extension of A acting in $\widetilde{\mathfrak{H}}$ ($\supset \mathfrak{H}$). Define the Straus extension*

$$T(\infty)=\left\{\{f_1,f_1'\}:\left\{\begin{pmatrix} f_1\\ 0\end{pmatrix},\begin{pmatrix} f_1'\\ f_2'\end{pmatrix}\right\}\in\widetilde{A},\ f_2'\in\mathfrak{H}_2\right\}$$

Let us say:
(i) $\widetilde{A}\in Naj_1(A)$ if $T(\infty)=\widetilde{A}$;
(ii) $\widetilde{A}\in Naj_2(A)$ if $T(\infty)=A$;
(iii) $\widetilde{A}\in Naj_2(A)$ if $A\neq T(\infty)\neq\widetilde{A}$.

Theorem 15 $\widetilde{A}\in Naj_2(A)\Leftrightarrow S_1=A\&\mathrm{mul}\,T_2=\{0\}$

Theorem 16 *Let $n_\pm(A)=\infty$. Then $\widetilde{A}\in Naj_2(A)$ iff $\tau(\cdot)\in R^s(\mathcal{H})$ and the function*

$$\tau^{(1)}(\lambda)=-(\tau(\lambda)-1/\lambda)^{-1}$$

satisfies the limit conditions in

$$\lim_{y\downarrow\infty} y^{-1}\tau(iy)=0, \tag{14}$$

$$\lim_{y\downarrow\infty} y\cdot\mathrm{Im}\,(\tau(iy)h,h)=\infty,\quad h\in\mathcal{H}\setminus\{0\}. \tag{15}$$

Example

Let A be a minimal differential operator in $L_2[0,1]$ associated with the differential expression $-D^2$ with the domain

$$\{f \in W_2^2[0,1] : f(0+) = f'(0+) = f(1) = f'(1) = 0\}.$$

Let the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be defined by

$$\Gamma_0 f = \begin{pmatrix} f(0+) \\ f(1) \end{pmatrix}, \quad \Gamma_1 f = \begin{pmatrix} f'(0+) \\ -f'(1) \end{pmatrix}.$$

Let $\widetilde{A}$ be a selfadjoint operator in $L_2[-1,1]$ associated with $-D^2$ and the periodic boundary conditions

$$f(1) = f(-1), \quad f'(1) = f'(-1).$$

Then S_2 is a minimal differential operator generated by $-D^2$ on the interval $[-1,0]$ and the induced boundary triplet $\{\mathbb{C}, \Gamma_0'', \Gamma_1''\}$ takes the form

$$\Gamma_0'' f = \begin{pmatrix} f(0-) \\ f(-1) \end{pmatrix}, \quad \Gamma_1'' f = \begin{pmatrix} -f'(0-) \\ f'(-1) \end{pmatrix}.$$

Then $\widetilde{A} \in Naj_2(A)$ since $S_1 = A$ and $\operatorname{mul} T_2 = \{0\}$.

Let $\widetilde{A}$ be a selfadjoint extension of A in $L_2(-\infty, 1]$ associated with $-D^2$ and the boundary condition

$$f'(1) = hf(1).$$

Then S_2 is a minimal differential operator generated in $L_2(-\infty, 1)$ by $-D^2$ and the induced boundary relation $\Gamma'' : S_2^* \to \mathbb{C}^2$ takes the form

$$\Gamma'' = \left\{ \left\{ \widehat{f}, \text{col } (f(0-), c, -f'(0-), hc) \right\} : \widehat{f} \in S_2^*, \, c \in \mathbb{C} \right\}.$$

Then $\widetilde{A} \in Naj_3(A)$ since $S_1 \neq A$. This fact can be illustrated also analitically, since the Weyl function, corresponding to Γ'' takes the form

$$\tau(\lambda) = \begin{pmatrix} i\sqrt{\lambda} & 0 \\ 0 & h \end{pmatrix}$$

and is not strict.

Admissibility

Corollary 17 *Given a a boundary triplet* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *the Kreĭn resolvent formula*

$$P_{\mathfrak{H}}(\widetilde{A}-\lambda)^{-1}\lceil\mathfrak{H} = (A_0-\lambda)^{-1} - \gamma(\lambda)\Big(M(\lambda)+\tau(\lambda)\Big)^{-1}\gamma(\bar{\lambda})^*$$

establishes a one-to-one correspondence between the set of Nevanlinna families $\tau(\cdot) \in \widetilde{R}(\mathcal{H})$ *and the set of minimal exit space selfadjoint extensions* $\widetilde{A} = \widetilde{A}^{(\tau)}$ *of* S*.*

Let A be densely defined. A family $\tau = \{\phi(\lambda), \psi(\lambda)\} \in R[\mathcal{H}]$ is called Π-admissible, if $\widetilde{A}^{(\tau)}$ is singlevalued.

Theorem 18 *Let* A *be a (nondensely defined) closed symmetric operator in* $\mathfrak{H}$ *with equal defect numbers* $n_+(A) = n_-(A) \leq \infty$, *let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be a boundary triplet for* A^* *with Weyl function* $M(\lambda)$, *and let* $\{\phi(\lambda), \psi(\lambda)\}$ *be a Nevanlinna pair in* $\mathcal{H}$*. Then the pair* $\{\phi(\lambda), \psi(\lambda)\}$ *is* Π*-admissible if and only if the following two conditions are satisfied:*

$$w - \lim_{y\uparrow\infty} \frac{\phi(\lambda)(\psi(iy) + M(iy)\phi(\lambda))^{-1}}{y} = 0 \qquad (16)$$

and

$$w - \lim_{y\uparrow\infty} \frac{\psi(\lambda)(\psi(iy) + M(iy)\phi(\lambda))^{-1}M(\lambda)}{y} = 0. \qquad (17)$$