A KREIN SPACE INTERPRETATION OF DIRAC OPERATORS

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The Standard Free Dirac Operator

To simplify the notation, we assume the mass $m = 1$ and the light speed $c = 1$.

The free Dirac operator is defined in the space $\mathcal{H} = L_2(\mathbb{R}^3; \mathbb{C}^4)$ identified with $\mathbb{C}^4 \otimes L_2(\mathbb{R}^3)$ as the following

$$H_0 = \sum_{j=1}^{3} \alpha_j \otimes D_j + \alpha_0 \otimes I_{L_2(\mathbb{R}^3)},$$

where $D_j = \mathrm{i}\partial/\partial x_j$ $(j = 1, 2, 3)$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, α_j $(j = 1, 2, 3, 4)$ are the Dirac matrices, i.e. 4×4 Hermitian matrices which satisfy the anticommutation relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk}, \quad j, k = 0, 1, 2, 3.$$

In the standard representation, the Dirac matrices α_j $(j = 0, 1, 2, 3)$ are chosen as follows

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \text{ for } j = 1, 2, 3; \ \alpha_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices ($\sigma_0 = I_2$ designates the 2×2 identity matrix).

We consider the operator H_0 defined on its maximal domain, i.e. on the Sobolev space $\mathrm{Dom}(H_0) = W_2^1(\mathbb{R}^3; \mathbb{C}^4)$. It is known that, on this domain,

- H_0 is a self-adjoint operator.
- The spectrum of H_0 is $(-\infty, -1] \cup [1, +\infty)$.
- H_0 has only continuous spectrum.

The Energy Space Representation (Friedrichs)

- Assume A is selfadjoint, $A = A^*$, and nonnegative, $\langle Ax, x\rangle \geq 0$ for all $x \in \mathrm{Dom}(A)$.

$$\mathrm{Dom}(A) \longrightarrow \mathrm{Dom}(A)/\ker(A) \longrightarrow \mathcal{K}_A$$

$$\Pi_A\colon\ \mathrm{Dom}(\Pi_A) = \mathrm{Dom}(A) \to \mathcal{K}_A$$

- $A^{1/2}$ is nonnegative selfadjoint, $\mathrm{Dom}(A^{1/2}) \supseteq \mathrm{Dom}(A)$ and $\mathrm{Dom}(A^{1/2})$ is a core for $A^{1/2}$.
- $\langle Ax, y\rangle = \langle A^{1/2}x, A^{1/2}y\rangle$ for all $x \in \mathrm{Dom}(A)$ and all $y \in \mathrm{Dom}(A^{1/2}$.
- The strong topology of $\mathcal{K}_A$ is given by the semi-norm $\|A^{1/2}\cdot\|$.

The Energy Space Representation for a Symmetric Operator

Let $\mathcal{H}$ be a (complex) Hilbert space, A a densely defined symmetric operator in $\mathcal{H}$, that is:

$$A\colon\ \mathrm{Dom}(A)(\text{ dense } \subseteq \mathcal{H}) \to \mathcal{H}$$
$$\langle Ax, y\rangle = \langle x, Ay\rangle,\ \text{ for all } x \in \mathrm{Dom}(A).$$

A pair $(\mathcal{K}, \Pi)$ is called a *Krein space induced by A* if:

(i) $\mathcal{K}$ is a Krein space;

(ii) Π is a linear operator with $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(\Pi) \subseteq \mathcal{H}$ and range in $\mathcal{K}$;

(iii) $\Pi\,\mathrm{Dom}(A)$ is dense in $\mathcal{K}$;

(iv) $[\Pi x, \Pi y]_{\mathcal{K}} = \langle Ax, y\rangle_{\mathcal{H}}$ for all $x \in \mathrm{Dom}(A)$ and all $y \in \mathrm{Dom}(\Pi)$.

- The pair $(\mathcal{K}, \Pi)$ is a Krein space induced by A if and only if:

(i) $\mathcal{K}$ is a Krein space;

(ii) Π is a linear operator with $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(\Pi) \subseteq \mathcal{H}$ and range in $\mathcal{K}$;

(iii) $\Pi \, \mathrm{Dom}(A)$ is dense in $\mathcal{K}$;

(iv)' $\Pi^\sharp \Pi \supseteq A$, in the sense $\Pi \, \mathrm{Dom}(A) \subseteq \mathrm{Dom}(\Pi^\sharp)$ and $\Pi^\sharp \Pi x = Ax$ for all $x \in \mathrm{Dom}(A)$.

- Without loss of generality, we can assume Π closed.

- A is bounded if Π is bounded, but not the other way.

Existence

Proposition *Let* A *be a densely defined and symmetric operator in a Hilbert space* $\mathcal{H}$. *The following assertions are equivalent:*

(a) *There exists a nonnegative quadratic form* q *on* $\mathrm{Dom}(A)$ *such that*

$$-q(x) \leq \langle Ax, x\rangle \leq q(x), \quad x \in \mathrm{Dom}(A).$$

(a)' *There exists a nonnegative operator* B *in* $\mathcal{H}$ *such that* $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(B)$ *and* $-\langle Bx, x\rangle_{\mathcal{H}} \leq \langle Ax, x\rangle_{\mathcal{H}} \leq \langle Bx, x\rangle_{\mathcal{H}}$ *for all* $x \in \mathrm{Dom}(A)$.

(b) *There exists a nonnegative quadratic form* q *on* $\mathrm{Dom}(A)$ *such that*

$$|\langle Ax, y\rangle|^2 \leq q(x)q(y), \quad x, y \in \mathrm{Dom}(A).$$

(b)' *There exists a nonnegative operator* B *in* $\mathcal{H}$ *such that* $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(B)$ *and* $|\langle Ax, y\rangle| \leq |\langle Bx, x\rangle|^{1/2}|\langle By, y\rangle|^{1/2}$ *for all* $x, y \in \mathrm{Dom}(A)$.

(c) $A \subseteq A_+ - A_-$ *for two nonnegative operators* $A_\pm$ *in* $\mathcal{H}$, *that is,* $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(A_+) \cap \mathrm{Dom}(A_-)$ *and* $Ax = A_+x - A_-x$ *for all* $x \in \mathrm{Dom}(A)$.

(d) *There exists a Kreĭn space induced by* A.

Corrolay For any densely defined symmetric operator A that admits a selfadjoint extension in $\mathcal{H}$, there exists a Kreĭn space induced by A.

Example Let A_- and A_+ be the differential operators on $L_2(\mathbb{R}_+)$ defined by the differential expressions

$$A_- = -\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2, \quad A_+ = -\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + 2\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + 1,$$

where $\mathrm{Dom}(A_+) = \mathrm{Dom}(A_-)$ is the Sobolev space $W_2^2(\mathbb{R}_+)$, with the Dirichlet boundary conditions at 0. Then both A_+ and A_- are nonnegative selfadjoint operators but the operator

$$A_+ - A_- = 2\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + 1$$

is a symmetric operator in $L_2(\mathbb{R}_+)$ with defect indices $(1, 0)$, and hence does not have selfadjoint extensions.

The Induced Kreĭn Space $(\mathcal{K}_A, \Pi_A)$

Let A be a selfadjoint operator in the Hilbert space $\mathcal{H}$. We consider the polar decomposition of A

$$A = S_A|A|, \quad |A| = (A^*A)^{1/2} = (A^2)^{1/2}, \quad S_A = \mathrm{sgn}(A)$$

$\mathrm{Dom}(A) = \mathrm{Dom}(|A|)$, and $|A|$ is a positive selfadjoint operator.

Let $\mathcal{K}_A = \mathcal{K}_{|A|}$. Recall that $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(|A|^{1/2})$ and that $\mathrm{Dom}(A)$ is a core for $|A|^{1/2}$. $\ker(S_A) = \ker(A)$, S_A leaves invariant $\mathrm{Dom}(A)$.

$$\mathrm{Dom}(A) = \mathcal{D}_+ \oplus \ker(A) \oplus \mathcal{D}_-$$

where $\mathcal{D}_\pm = \mathrm{Dom}(A) \cap \ker(S_A \mp I)$.

We complete $\mathcal{D}_\pm$ with respect to the norm $\||A|^{1/2}\cdot\|$ to $\mathcal{K}_A^\pm$,

$$\mathcal{K}_A = \mathcal{K}_A^+ \oplus \mathcal{K}_A^- = \mathcal{K}_A^+[+]\mathcal{K}_A^-$$

yields a Kreĭn space $(\mathcal{K}_A; [\cdot,\cdot])$.

Let Π_A be the operator which is obtained by composing the canonical surjection $\mathrm{Dom}(A) \to \mathrm{Dom}(A)/\ker(A)$ with the embedding of $\mathrm{Dom}(A)/\ker(A)$ into its Hilbert space completion $\mathcal{K}_{|A|} = \mathcal{K}_A$.

Proposition If A is a selfadjoint operator on the Hilbert space $\mathcal{H}$ then, with the notation as before, $(\mathcal{K}_A, \Pi_A)$ is a Kreĭn space induced by A.

The Lifting Theorem

Theorem Let A and B be selfadjoint operators in Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and let $(\mathcal{K}_A, \Pi_A)$ and $(\mathcal{K}_B, \Pi_B)$ be their Krein spaces induced by A and B, respectively.

Let $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $S \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ be such that

$$[Bx, Ty] = [Sx, Ay], \text{ for all } x \in \mathrm{Dom}(B),\ y \in \mathrm{Dom}(A).$$

Then there exist uniquely $\widetilde{T} \in \mathcal{B}(\mathcal{K}_A, \mathcal{K}_B)$ and $\widetilde{S} \in \mathcal{B}(\mathcal{K}_B, \mathcal{K}_A)$ such that:

$$\widetilde{T}\Pi_A = \Pi_B T x \text{ for all } x \in \mathrm{Dom}(A)$$

$$\widetilde{S}\Pi_B y = \Pi_A S y \text{ for all } y \in \mathrm{Dom}(B)$$

$$[\widetilde{S}h, k]_{\mathcal{K}_B} = [h, \widetilde{T}k]_{\mathcal{K}_A} \text{ for all } h \in \mathcal{K}_B,\ k \in \mathcal{K}_A.$$

- For all $x \in \mathrm{Dom}(A)$ we have $Tx \in \mathrm{Dom}(B)$ and $BTx = S^*Ax$.

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- For all $h \in \mathrm{Dom}(A)$ and all $n \in \mathbb{N}$ we have $(ST)^n h \in \mathrm{Dom}(A)$ and $A(ST)^n h = (T^* S^*)^n Ah$.

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$$\langle BTx, Tx\rangle \le r(ST)\langle Ax, x\rangle$$

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- For all $x \in \mathrm{Dom}(A)$ we have

$$\|B^{1/2}Tx\| \leq \sqrt{r(ST)}\|A^{1/2}x\|$$

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For all $n \geq 1$ and all $x \in \mathrm{Dom}(A)$

$$\|B^{\frac{1}{2}}Tx\|^2 \leq \|(ST)^{2^n}\|^{\frac{1}{2^n}} \, \|A^{\frac{1}{2}}x\|^{2(1-\frac{1}{2^n})} \, \|Ax\|^{\frac{1}{2^n}} \, \|x\|^{\frac{1}{2^n}}.$$

Spectral Preservation

Let A be a nonnegative selfadjoint operator on the Hilbert space $\mathcal{H}$, and let $(\mathcal{K}_A, \Pi_A)$ be the Hilbert space induced by A in the energy space representation.

Corollary If $T \in \mathcal{B}(\mathcal{H})$ satisfies the condition

$$\langle Ax, Ty \rangle = \langle Tx, Ay \rangle, \quad x, y \in \mathrm{Dom}(A)$$

then T can be lifted to a bounded operator $\widetilde{T} \in \mathcal{B}(\mathcal{K}_A)$ and

- $\sigma(\widetilde{T}) \subseteq \sigma(T)$.

- If λ belongs to the discrete spectrum of T then λ belongs to the discrete spectrum of $\widetilde{T}$ and their corresponding root subspaces are the same.

- If T has only discrete spectrum, then $\widetilde{T}$ has only discrete spectrum and $\sigma(T) = \sigma(\widetilde{T})$.

- If T is compact then $\widetilde{T}$ is compact on $\mathcal{K}_A$.

Let A be a nonnegative selfadjoint operator on the Hilbert space $\mathcal{H}$, and let $(\mathcal{K}_A, \Pi_A)$ be the Hilbert space induced by A in the energy space representation.

Corollary Let T be a selfadjoint (unbounded) linear operator in $\mathcal{H}$ such that for all $\zeta \in \mathbb{C} \setminus \mathbb{R}$ and all $x \in \mathrm{Dom}(A)$ we have

$$(\zeta I - T)^{-1} A x = A(\zeta I - T)^{-1} x.$$

Then

- T can be lifted to a linear operator $\widetilde{T}$ in $\mathcal{K}_A$.
- $\widetilde{T}$ is selfadjoint operator in the Hilbert space $\mathcal{K}_A$.
- $\sigma(\widetilde{T}) \subseteq \sigma(T)$.

Uniqueness

Theorem Let A be a selfadjoint operator in the Hilbert space $\mathcal{H}$. The following statements are equivalent:

(i) The Kreĭn space induced by A is unique, modulo unitary equivalence.

(ii) A has a lateral spectral gap, that is, there exists an $\epsilon > 0$ such that either $(0, \epsilon) \subset \rho(A)$ or $(-\epsilon, 0) \subset \rho(A)$.

(iii) For some (equivalently, for any) Kreĭn space $(\mathcal{K}, \Pi)$ induced by A, the linear manifold $\Pi \operatorname{Dom}(A)$ contains a maximal uniformly definite subspace of $\mathcal{K}$.

Representations in Terms of the Canonical Mapping Π

Proposition *Let $T \in \mathcal{C}(\mathcal{H}, \mathcal{H}_1)$, that is, T is a closed linear operator with domain $\mathrm{Dom}(T)$ dense in the Hilbert space $\mathcal{H}$ and range $\mathrm{Ran}(T)$ in the Hilbert space $\mathcal{H}_1$ such that for some $c > 0$ we have*

$$\|Tu\|_{\mathcal{H}_1} \geq c\|u\|_{\mathcal{H}}, \quad u \in \mathrm{Dom}(T). \tag{1}$$

Let also J be a symmetry on $\mathrm{Ran}(T)$.

Then, the operator $A = T^*JT$ is selfadjoint that has a spectral gap in the neighbourhood of 0, and $(\mathrm{Ran}(T), T)$ is a Kreĭn space induced by A.

The Free Dirac Operator

The free Dirac operator is defined in the space $\mathcal{H} = L_2(\mathbb{R}^3; \mathbb{C}^4)$ identified with $\mathbb{C}^4 \otimes L_2(\mathbb{R}^3)$ as the following

$$H_0 = \sum_{j=1}^{3} \alpha_j \otimes D_j + \alpha_0 \otimes I_{L_2(\mathbb{R}^3)},$$

where $D_j = \mathrm{i}\partial/\partial x_j$ $(j = 1, 2, 3)$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, α_j $(j = 1, 2, 3, 4)$ are the Dirac matrices, i.e. 4×4 Hermitian matrices which satisfy the anticommutation relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk}, \quad j, k = 0, 1, 2, 3.$$

In the standard representation, the Dirac matrices α_j $(j = 0, 1, 2, 3)$ are chosen as follows

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \text{ for } j = 1, 2, 3; \ \alpha_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices ($\sigma_0 = I_2$ designates the 2×2 identity matrix).

Note that by applying the Fourier transformation to the elements of the space $L_2(\mathbb{R}^3;\mathbb{C}^4)$ the operator H_0 is transformed (in the momentum space) into a multiplication operator by the following matrix-valued function

$$h_0(\xi) = \begin{bmatrix} \sigma_0 & \sigma(\xi) \\ \sigma(\xi) & -\sigma_0 \end{bmatrix}$$

where

$$\sigma(\xi) = \xi_1\sigma_1 + \xi_2\sigma_2 + \xi_3\sigma_3,\ \ \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.$$

The Fourier transformation is defined by the formula

$$\hat{u}(\xi) = (Fu)(\xi) = \frac{1}{(2\pi)^{3/2}} \int u(x) e^{\mathrm{i}\langle x,\xi\rangle} \mathrm{d}x,\ \ u \in L_2(\mathbb{R}^3)$$

in which $\langle x, \xi\rangle$ designates the scalar product of the elements $x, \xi \in \mathbb{R}^3$ (here and in what follows $\int := \int_{\mathbb{R}^3}$).

The matrix $h_0(\xi)$ is the symbol of the operator H_0 considered as a matrix differential operator with constant coefficients. This matrix has the following eigenvalues

$$\lambda_1(\xi) = \lambda_2(\xi) = r(\xi),\ \lambda_3(\xi) = \lambda_4(\xi) = -r(\xi),$$

where $r(\xi) = (1 + |\xi|^2)^{1/2}$.

The unitary transformation $U(\xi)$ which brings $h_0(\xi)$ to the diagonal form is given explicitly by

$$U(\xi) = \begin{bmatrix} a(\xi)I_2 & -b(\xi)\sigma(\xi) \\ b(\xi)\sigma(\xi) & -a(\xi)I_2 \end{bmatrix},$$

where $a(\xi) = (\frac{1}{2}(1 + r(\xi))^{-1})^{1/2}$ and $b(\xi) = a(\xi)(1 + \gamma(\xi))^{-1}$. Thus, we have

$$U(\xi)h_0(\xi)U(\xi)^* = \alpha_0 r(\xi).$$

Now, we let

$$T(\xi) = r(\xi)^{\frac{1}{2}} U(\xi),$$

and denote by $T = T(D)$ the pseudodifferential operator corresponding to its symbol $T(\xi)$. The operator T is defined in the space $\mathcal{H} = L_2(\mathbb{R}^3; \mathbb{C}^4)$ by

$$(Tu)(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int T(\xi)\hat{u}(\xi) e^{-i\langle x, \xi\rangle} \mathrm{d}\xi, \; x \in \mathbb{R}^n,$$

on the domain $\mathrm{Dom}(T) = \{u \in L_2(\mathbb{R}^3; \mathbb{C}^4) : T(\xi)\hat{u}(\xi) \in L_2(\mathbb{R}^3; \mathbb{C}^4)\}$.

Obviously, $u \in \mathrm{Dom}(T)$ if and only if $\hat{u} \in L_{2,r}(\mathbb{R}^3; \mathbb{C}^4)$, where $L_{2,r}(\mathbb{R}^3; \mathbb{C}^4)$ stands for the space weighted by $r(\xi) = (1 + |\xi|^2)^{\frac{1}{2}}$, i.e. the space all functions $f \in L_2(\mathbb{R}^3; \mathbb{C}^4)$ such that $rf \in L_2(\mathbb{R}^3; \mathbb{C}^4)$. Note that $F^* L_{2,r}(\mathbb{R}^3; \mathbb{C}^4) = W_2^1(\mathbb{R}^3; \mathbb{C}^4)$ (the Fourier transformation in the space $L_2(\mathbb{R}^3; \mathbb{C}^4)$ is again denoted by F).

It follows that the factorization

$$H_0 = T^*(\alpha_0 \otimes I_{L_2(\mathbb{R}^3)})T.$$

Since

$$\|Tu\|^2 = \int |T(\xi)\hat{u}(\xi)|^2 \mathrm{d}\xi = \int r(\xi)|\hat{u}(\xi)|^2 \mathrm{d}\xi \geq \int |\hat{u}(\xi)|^2 \mathrm{d}\xi = \|u\|$$

for all $u \in \mathrm{Dom}(T)$, the condition from Proposition is fulfilled. In particular, the range $\mathrm{Ran}(T)$ is closed in the space $L_2(\mathbb{R}^3; \mathbb{C}^4)$, and so we have the Hilbert space

$$\mathcal{G}_T = (\mathrm{Ran}(T), \| \cdot \|_{L_2(\mathbb{R}^3; \mathbb{C}^4)}).$$

On the space $\mathcal{H} = L_2(\mathbb{R}^3; \mathbb{C}^4)$ we consider the symmetry given by

$$Ju = \alpha_0 \otimes I_{L_2(\mathbb{R}^r)} u, \quad u \in \mathcal{L}_2(\mathbb{R}^r), \tag{2}$$

and hence the Hilbert space $\mathcal{G}_T$ equipped with the indefinite scalar product defined by J becomes a Kreĭn space that we denote by $\mathcal{K}$. We have the decomposition

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-,$$

where the orthogonal projection operators from $\mathcal{K}$ onto $\mathcal{K}_\pm$ are given by

$$P_\pm = \frac{1}{2}(I \pm \alpha_0) \otimes I_{L_2(\mathbb{R}^r)}.$$

We conclude that the pair $(\mathcal{K}, \Pi)$, where $\Pi = T$ (recall that T is the pseudodifferential operator defined in the space $L_2(\mathbb{R}^3; \mathbb{C}^4)$) is a Kreĭn space induced by the free Dirac operator H_0, by the previous Proposition.