Block Numerical Ranges of Nonnegative Matrices

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joint work with K.-H. Förster

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For a nonnegative matrix A with index of imprimitivity m we have

$$\left\{\lambda \in W_{\mathfrak{D}}(A) : |\lambda| = w_{\mathfrak{D}}(A)\right\} = \left\{w_{\mathfrak{D}}(A)e^{2k\pi i/m}, k = 0, 1, \ldots, m-1\right\}.$$

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$$C = \zeta D A D^{-1}$$

for some $\ell \times \ell$ square matrix D with $|D| = \mathbf{I}_{\mathbb{C}^\ell}$.

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