What spectra can non-self-adjoint Sturm-Liouville operators have?

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- every EV is geometrically simple (the nullspace of $T - \lambda$ is of dim ≤ 1) and algebraically simple (no Jordan blocks of size $\geq 1 \Longleftrightarrow$ no solution to $(T-\lambda)y_0 = 0$, $(T-\lambda)y_1 = y_0$)

Known results (s-a case)

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Theorem A. *$\lambda_1 < \lambda_2 < \cdots$ and $\mu_1 < \mu_2 < \cdots$ are Dirichlet resp. Neumann-Dirichlet spectra of a SL expression* (1) *with $q \in L_2(0,1)$ iff these sequences interlace (i.e., $\mu_n < \lambda_n < \mu_{n+1}$, $\forall n \in \mathbb{N}$) and obey*

$$\lambda_n = \pi^2 n^2 + A + a_n, \tag{3}$$

$$\mu_n = \pi^2 (n - \tfrac{1}{2})^2 + A + b_n \tag{4}$$

with $A \in \mathbb{R}$ and $(a_n), (b_n) \in \ell_2$.

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Problems: EV's might be non-real and/or non-simple!

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Cor.: $\forall (\lambda_1, \lambda_2, \dots, \lambda_N) \in \mathbb{C}^N$ s.t. $\lambda_k = \lambda_l$ for no k, l and $\forall (m_1, m_2, \dots, m_N) \in \mathbb{N}^N$ $\exists$ a SL operator (1) s.t. $\lambda_1, \lambda_2, \dots, \lambda_N$ are its Dirichlet EV's of (algebraic) multiplicities $m_1, m_2, \dots, m_N$ resp.

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Neither Thm B nor Cor implies that **any** complex sequence (λ_n) obeying (3) is the spectrum of some SL operator (1)–(2)!

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- give criterion for solubility and reconstruction algorithm

Singular SL operators

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$$Ty = T(q)y = l(q)(y) := -(y' - \sigma y)' - \sigma y'$$

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Example 1: $q = \alpha\delta(\cdot - \frac{1}{2})$. Take

$$\sigma(x) = 0 \quad \text{for } x \le \tfrac{1}{2}, \qquad \sigma(x) = \alpha \quad \text{for } x > \tfrac{1}{2}$$

then $l(q)(y) = -y''$ if $x \neq \frac{1}{2}$ and $y \in \mathrm{dom}\, T$ means y is continuous at $x = \frac{1}{2}$ and $y'(\frac{1}{2}+) - y'(\frac{1}{2}-) = \alpha y(\frac{1}{2})$.

Example 2: $q = (x - \frac{1}{2})^{-1}$. Restriction-extension theory defines the corresponding (non-s.a.) operators T_γ, $\gamma \in \mathbb{C} \cup \{\infty\}$ by the interface conditions $y(\frac{1}{2}+) = y(\frac{1}{2}-) =: y(\frac{1}{2})$, $y'(\frac{1}{2}+) - y'(\frac{1}{2}-) = \gamma y(\frac{1}{2})$; cf. Kurasov (1996), Bodenstorfer a.o. (2000). This corresponds to

$$\sigma(x) = \begin{cases} \log(\frac{1}{2} - x) & \text{for } x \leq \frac{1}{2}, \\ \log(x - \frac{1}{2}) + \gamma & \text{for } x > \frac{1}{2}. \end{cases}$$

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Theorem 1. *For any sequence* $(\lambda_n)_{n\in\mathbb{N}}$ *of complex numbers satisfying* (5) $\exists q \in W_2^{-1}(0,1)$ *s.t. the spectrum of the SL operator* $T(q,\infty)$ *coincides with* (λ_n).

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Similarly, for any sequence $(\mu_n)_{n\in\mathbb{N}}$ *of complex numbers satisfying* (6) $\exists q \in W_2^{-1}(0,1)$ *and* $h \in \mathbb{C}$ *s.t. the spectrum of* $T(q,h)$ *coincides with* (μ_n).

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Theorem 2. *Assume that sequences* $(\lambda_n)_{n\in\mathbb{N}}$ *and* $(\mu_n)_{n\in\mathbb{N}}$ *of complex numbers verify* (5) *and* (6) *respectively.*

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(b) there are $q \in W_2^{-1}(0,1)$ *and* $h \in \mathbb{C}$ *such that the sequences* $(\lambda_n)_{n\in\mathbb{N}}$ *and* $(\hat\mu_n)_{n\in\mathbb{N}}$ *are spectra of the Sturm–Liouville operators* $T(q,\infty)$ *and* $T(q,h)$ *respectively.*

We remark that this theorem does not answer the question whether for any two disjoint finite sequences in $\mathbb{C}$ there are $q \in W_2^{-1}(0,1)$ and $h \in \mathbb{C}$ such that the first sequence is in the spectrum of $T(q,\infty)$ and the second in that of $T(q,h)$.

Reconstruction from norming constants

In the s-a case, the norming constants are

$$\alpha_n := \int_0^1 |y(x, \lambda_n)|^2 \, dx,$$

where $y(\cdot, \lambda)$ solves $l(q)y = \lambda y$ with the initial conditions $y(0) = 0, y^{[1]} = \sqrt{\lambda}$ if $h = \infty$ and $y(0) = 1, y^{[1]}(0) = h$ otherwise.

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Theorem C. *For any sequences of real numbers* $(\lambda_n)_{n\in\mathbb{N}}$ *and* $(\alpha_n)_{n\in\mathbb{N}}$ *s.t.* λ_n *strictly increase and obey* (5) *and* α_n *are positive and obey* (7)*, there is a real-valued* $q \in W_2^{-1}(0,1)$ *s.t.* λ_n *and* α_n *are resp. the EV's and norming constants of the SL operator* $T(q,\infty)$.

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Similarly, if in the above assumptions (λ_n) *is replaced by a sequence* (μ_n) *obeying* (6) *instead of* (5)*, then the conclusion holds with a SL operator* $T(q,h)$ *for some* $q \in W_2^{-1}(0,1)$ *and* $h \in \mathbb{R}$*.*

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- Can one reconstruct a SL operator from the spectrum and norming constants?

Norming constants in the non-s-a case

Assume that λ is an eigenvalue of $T(q,h)$ of algebraic multiplicity m; then $\lambda = \lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+m-1}$ for some $n \in \mathbb{N}$. Introduce the functions $y_n, \dots, y_{n+m-1}$ via

$$y_{n+j}(x) := \frac{1}{j!}\frac{\partial^j y(x,z)}{\partial z^j}\Big|_{z=\lambda_n}, \qquad j = 0, 1, \dots, m-1.$$

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Then y_n is an EF of $T(q,h)$ corresponding to the EV λ_n, $y_{n+j} \in \operatorname{dom} T(q,h)$, and

$$T(q,\infty) y_{n+j} = \lambda_n y_{n+j} + y_{n+j-1},$$

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In this way we construct the sequence $(y_k)_{k\in\mathbb{N}}$, in which y_k is an eigen- or associated function of $T(q,h)$ corresponding to the eigenvalue λ_k.

Norming constants (cont'd)

Put

$$a_{kl} := \langle y_k, y_l \rangle,$$

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We call the number α_k, $k \in \mathbb{N}$, the norming constant corresponding to the eigenvalue λ_k.

Reconstruction from norming constants

If λ_k is a simple eigenvalue, then $\alpha_k = \int_0^1 y_k^2(x)\,dx$, which agrees with the above definition in the self-adjoint case.

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A similar statement holds if instead of the asymptotics (5) *that of* (6) *is assumed, resulting in a SL operator $T(q,h)$ with some $q \in W_2^{-1}(0,1)$ and $h \in \mathbb{C}$.*

Regular potentials

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Corollary. *Assume that sequences* $(\lambda_n)_{n\in\mathbb{N}}$, $(\mu_n)_{n\in\mathbb{N}}$, *and* $(\alpha_n)_{n\in\mathbb{N}}$ *of complex numbers are such that*

$$\lambda_n = \pi^2 n^2 + A + \tilde\lambda_n,$$
$$\mu_n = \pi^2 (n - \tfrac{1}{2})^2 + B + \tilde\mu_n,$$
$$\alpha_n = \frac{1}{2} + \frac{\tilde\alpha_n}{n}$$

for some complex A *and* B *and some complex* ℓ_2*-sequences* $(\tilde\lambda_n)$, $(\tilde\mu_n)$, *and* $(\tilde\alpha_n)$. *Then the conclusions of Theorems 1–3 hold true with a complex-valued* $q \in L_2(0,1)$.

Thank you!