The inverse problem for Kreĭn orthogonal entire matrix functions

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Continuous analogues of orthogonal polynomials

$k \in L_1^{n\times n}[-\omega, \omega]$

$k(t)^* = k(-t), \ -\omega \le t \le \omega$ [hermitian]

Let $f \in L_1^{n\times n}[0, \omega]$ be a solution of

$$(1) \quad f(t) - \int_0^\omega k(t-s) f(s)\, ds = k(t), \ 0 \le t \le \omega.$$

Put

$$(2) \qquad \Phi(\lambda) = I + \int_0^\omega e^{i\lambda t} f(t)\, dt, \ \lambda \in \mathbb{C}.$$

For $n = 1$ functions of type (2) have been introduced by M.G. Kreĭn as continuous analogues of the classical Szegö orthogonal polynomials with respect to the unit circle. For this reason Φ is referred to as a *Kreĭn orthogonal function* (KOF) generated by k and ω.

N.B.: There is at most one KOF generated by k and ω.

Inverse problem for scalar functions

For $n = 1$, Kreĭn in the fifties and Kreĭn and Langer in the eighties proved a number of remarkable results. One of these results is the solution to the inverse problem. Let

$$\Phi(\lambda) = I + \int_0^\omega e^{i\lambda t} f(t)\, dt, \ f \in L_1[0, \omega].$$

THM. *The entire function Φ is a KOF if and only if Φ has no real zeroes and no conjugate pairs of zeroes.*

This talk concerns the matrix-valued analogue of this theorem.

Root functions

Let $M(\lambda)$ be an entire $n \times n$ matrix function, and assume that $\det M(\lambda) \not\equiv 0$. A $\mathbb{C}^n$-valued function φ, analytic in a neighborhood of $\lambda_0 \in \mathbb{C}$, is called a *root function* of $M(\lambda)$ at λ_0 of *order at least* m (≥ 1) if

(i) $\varphi(\lambda_0) \neq 0$,

(ii) $M(\lambda)\varphi(\lambda)$ has a zero at λ_0 of order

at least m, that is,

$$M(\lambda)\varphi(\lambda) = \sum_{\nu=m}^{\infty} (\lambda - \lambda_0)^{\nu} y_{\nu}.$$

In particular, $M(\lambda_0)\varphi(\lambda_0) = 0$, and hence by (i) we have $\det M(\lambda_0) = 0$. Conversely, if $\det M(\lambda_0) = 0$, then $M(\lambda)$ has a root function at λ_0 of order at least 1.

- *Canonical systems of root functions*

- *Jordan chains*

Main theorem

Let Φ be the entire matrix function

$$\Phi(\lambda) = I + \int_0^\omega e^{i\lambda t} f(t)\,dt,\ f \in L_1^{n\times n}[0,\omega].$$

Main Theorem. *The function Φ is a KOF if and only if* $\det \Phi(\lambda)$ *has no real zero and for any symmetric pair of zeros $\lambda_0, \bar{\lambda}_0$ of* $\det \Phi(\lambda)$ *the following condition* $C(\lambda_0)$ *is fulfilled.*

Condition $C(\lambda_0)$: If φ is a root function of $\Phi(\lambda)$ at λ_0 and ψ is a root function of $\Phi(\lambda)$ at $\bar{\lambda}_0$, both of order at least m, then the function $\psi(\bar{\lambda})^* \varphi(\lambda)$ has a zero at λ_0 of order at least m.

Comments: (1) In the scalar case condition $C(\lambda_0)$ is never fulfilled. (2) When Φ is a KOF, how to find the kernel function corresponding to Φ?

An associate matrix function equation

Let Φ be the entire matrix function

$$\Phi(\lambda) = I + \int_0^\omega e^{i\lambda t} f(t)\, dt,\ f \in L_1^{n\times n}[0,\omega].$$

With Φ we associate the following matrix function equation

$$\text{(AE)} \quad U(\lambda)\Phi(\lambda) + \Phi(\bar{\lambda})^* V(\lambda) = I_n,\ \lambda \in \mathbb{C}.$$

The unknowns U, V are required to be entire $n \times n$ matrix functions of a special form, namely

$$\text{(a)} \quad U(\lambda) = I_n + \int_0^\omega e^{i\lambda t} u(t)\, dt,\ u \in L_1^{n\times n}[0,\omega],$$

$$\text{(b)} \quad V(\lambda) = \int_\omega^{2\omega} e^{i\lambda t} v(t)\, dt,\ v \in L_1^{n\times n}[\omega, 2\omega].$$

If U, V given by (a) and (b) satisfy (AE), then U, V is called a *regular solution pair* of the associate equation (AE).

Reducing the inverse problem

Let Φ be the entire matrix function

$$\Phi(\lambda) = I + \int_0^\omega e^{i\lambda t} f(t)\, dt, \; f \in L_1^{n\times n}[0,\omega].$$

THM 1. *The function Φ is a KOF if and only if the associated equation (AE) has a regular solution pair. Moreover, given any regular solution pair U, V of (AE), a hermitian matrix function $k \in L_1^{n\times n}[-\omega,\omega]$ such that Φ is a KOF generated by k can be obtained as the restriction to the interval $[-\omega,\omega]$ of the inverse Fourier transform of the matrix function*

$$\Big(U(\bar{\lambda})^* - V(\lambda)\Big)\Phi(\lambda)^{-1}, \quad \lambda \in \mathbb{R}.$$

In other words, $k = \ell|_{[-\omega,\omega]}$ where

$$\int_{-\infty}^{\infty} e^{i\lambda s} \ell(s)\, ds = \Big(U(\bar{\lambda})^* - V(\lambda)\Big)\Phi(\lambda)^{-1}.$$

The proof is “algebraic” in flavor using band method arguments.

Conclusion from Theorem 1

Let Φ be the entire matrix function

$$\Phi(\lambda) = I + \int_0^\omega e^{i\lambda t} f(t)\,dt,\ f \in L_1^{n\times n}[0,\omega].$$

Consider the associate equation:

(AE) $U(\lambda)\Phi(\lambda) + \Phi(\bar{\lambda})^* V(\lambda) = I_n, \quad \lambda \in \mathbb{C}.$

To prove the main theorem it suffices to show that the following two statements are equivalent.

(α) Equation (AE) has a regular solution pair U, V.

(β) The function $\det \Phi(\lambda)$ has no real zero and for any conjugate pair of zeros $\lambda_0, \bar{\lambda}_0$ of $\det \Phi(\lambda)$ condition C(λ_0) is fulfilled.

The problem to prove the main theorem is now reduced to a problem about entire matrix function equations.

Rewriting the associate equation

$$\Phi(\lambda) = I + \int_0^\omega e^{i\lambda t} f(t)\,dt,\ f \in L_1^{n\times n}[0,\omega]$$

$$\text{(AE)} \quad U(\lambda)\Phi(\lambda) + \Phi(\bar{\lambda})^* V(\lambda) = I_n,\ \lambda \in \mathbb{C}.$$

Introduce:

$$\mathcal{B}(\lambda) = I_n + \int_{-\omega}^0 e^{i\lambda t}\beta(t)dt,\ \beta(t) = f(-t),$$

$$\mathcal{D}(\lambda) = I_n + \int_0^\omega e^{i\lambda t}\delta(t)\,dt,\ \delta(t) = f(t)^*,$$

$$X(\lambda) = \int_0^\omega e^{i\lambda t} x(t)\,dt,\ x(t) = u(\omega - t),$$

$$Y(\lambda) = \int_{-\omega}^0 e^{i\lambda t} y(t)\,dt,\ y(t) = v(\omega - t),$$

$$G(\lambda) = e^{i\lambda\omega} I_n - e^{i\lambda\omega}\mathcal{B}(\lambda).$$

Replacing λ by $-\lambda$, (AE) is equivalent to

$$X(\lambda)\mathcal{B}(\lambda) + \mathcal{D}(\lambda)Y(\lambda) = G(\lambda),\ \lambda \in \mathbb{C}.$$

A class of matrix function equations

Consider the entire matrix function equation

(E) $\quad X(\lambda)\mathcal{B}(\lambda) + \mathcal{D}(\lambda)Y(\lambda) = G(\lambda), \ \lambda \in \mathbb{C}.$

The coefficients $\mathcal{B}$ and $\mathcal{D}$ are given,

$$\mathcal{B}(\lambda) = I_n + \int_{-\omega}^{0} e^{i\lambda t} b(t)\, dt, \ b \in L_1^{n\times n}[-\omega, 0],$$

$$\mathcal{D}(\lambda) = I_n + \int_{0}^{\omega} e^{i\lambda t} d(t)\, dt, \ d \in L_1^{n\times n}[0, \omega].$$

The right hand side G is known,

$$G(\lambda) = \int_{-\omega}^{\omega} e^{i\lambda t} g(t)\, dt, \ g \in L_1^{n\times n}[-\omega, \omega].$$

Problem: Find entire $n \times n$ matrix functions X and Y satisfying (E) and

$$X(\lambda) = \int_{0}^{\omega} e^{i\lambda t} x(t)\, dt, \ x \in L_1^{n\times n}[0, \omega],$$

$$Y(\lambda) = \int_{-\omega}^{0} e^{i\lambda t} y(t)\, dt, \ y \in L_1^{n\times n}[-\omega, 0].$$

Matrix function equations – continued

(E) $\quad X(\lambda)\mathcal{B}(\lambda)+\mathcal{D}(\lambda)Y(\lambda)=G(\lambda),\ \lambda\in\mathbb{C}.$

Condition SC(λ_0): If φ and ψ are root functions of $\mathcal{B}(\lambda)$ and $\mathcal{D}(\lambda)^\top$, respectively, both at λ_0 and of order at least m, then the function $\psi(\lambda)^\top G(\lambda)\varphi(\lambda)$ has a zero at λ_0 of order at least m.

THM 2. *In order that there exist functions X, Y satisfying equation* (E) *of the form*

$$X(\lambda)=\int_0^\omega e^{i\lambda t}x(t)\,dt,\ x\in L_1^{n\times n}[0,\omega],$$

$$Y(\lambda)=\int_{-\omega}^0 e^{i\lambda t}y(t)\,dt,\ y\in L_1^{n\times n}[-\omega,0],$$

it is necessary and sufficient that for each common zero λ_0 of $\det\mathcal{B}(\lambda)$ *and* $\det\mathcal{D}(\lambda)$ *condition* SC(λ_0) *is satisfied.*

Comments: (1) When (E) is derived from (AE), C(λ_0) $\Rightarrow$ SC(λ_0). (2) Necessity is obvious.

About the proof of THM 2

To prove THM 2 we rewrite equation (E) as

$$\mathbf{r}\Big(X(\lambda)\Big)\Big(\mathcal{B}(\lambda)\otimes I_n\Big)+ \\ +\mathbf{r}\Big(Y(\lambda)\Big)\Big(I_n\otimes\mathcal{D}(\lambda)\Big)=\mathbf{r}\Big((G(\lambda)\Big).$$

Here $\otimes$ is the (right) Kronecker product, and for a $k\times\ell$ matrix M we have

$$\mathbf{r}(M)=[m_{11}\ \cdots\ m_{k1}\ m_{12}\ \cdots\ m_{k2}\ \cdots \\ \cdots\ m_{1\ell}\ \cdots\ m_{k\ell}].$$

The map $\mathbf{r}$ is bijective from $\mathbb{C}^{k\times\ell}$ onto $\mathbb{C}^{1\times k\ell}$.

LEM. *Let $\lambda_0\in\mathbb{C}$, and assume that condition SC(λ_0) is satisfied. Then each common root function Φ of $\mathcal{B}(\lambda)\otimes I_n$ and $I_n\otimes\mathcal{D}(\lambda)^\top$ at λ_0, in both cases of order at least m, is a root function of $\mathbf{r}\Big((G(\lambda)\Big)$ at λ_0 of order at least m.*

Comment: $\mathcal{B}(\lambda)\otimes I_n$ and $I_n\otimes\mathcal{D}(\lambda)^\top$ commute and have similar positions in the equation.

Another class of matrix function equations

Consider the entire matrix function equation

(F) $\quad X(\lambda)\mathcal{B}(\lambda) + Y(\lambda)\mathcal{D}(\lambda) = G(\lambda), \ \lambda \in \mathbb{C}.$

The coefficients $\mathcal{B}$ and $\mathcal{D}$ are given,

$$\mathcal{B}(\lambda) = I_n + \int_{-\omega}^{0} e^{i\lambda t} b(t)\, dt, \ b \in L_1^{n\times n}[-\omega, 0],$$

$$\mathcal{D}(\lambda) = I_n + \int_{0}^{\omega} e^{i\lambda t} d(t)\, dt, \ d \in L_1^{n\times n}[0, \omega].$$

The right hand side G is known,

$$G(\lambda) = \int_{-\omega}^{\omega} e^{i\lambda t} g(t)\, dt, \ g \in L_1^{n\times n}[-\omega, \omega].$$

Problem: Find entire $n \times n$ matrix functions X and Y satisfying (F) and

$$X(\lambda) = \int_{0}^{\omega} e^{i\lambda t} x(t)\, dt, \ x \in L_1^{n\times n}[0, \omega],$$

$$Y(\lambda) = \int_{-\omega}^{0} e^{i\lambda t} y(t)\, dt \ y \in L_1^{n\times n}[-\omega, 0].$$

(F) $X(\lambda)\mathcal{B}(\lambda) + Y(\lambda)\mathcal{D}(\lambda) = G(\lambda), \ \lambda \in \mathbb{C}.$

Condition TC(λ_0): If φ is a common root function of $\mathcal{B}(\lambda)$ and $\mathcal{D}(\lambda)$ at λ_0, in both cases of order at least m, then the function $G(\lambda)\varphi(\lambda)$ has a zero at λ_0 of order at least m.

THM 3. *Assume $\mathcal{B}(\lambda)$ and $\mathcal{D}(\lambda)$ commute for each $\lambda \in \mathbb{C}$. There exist functions X, Y satisfying equation* (F) *and of the form*

$$X(\lambda) = \int_0^\omega e^{i\lambda t} x(t)\, dt, \ x \in L_1^{n\times n}[0,\omega],$$

$$Y(\lambda) = \int_{-\omega}^0 e^{i\lambda t} y(t)\, dt, \ y \in L_1^{n\times n}[-\omega,0],$$

if and only if condition TC(λ_0) *is satisfied for each common zero λ_0 of* $\det \mathcal{B}(\lambda)$ *and* $\det \mathcal{D}(\lambda)$.

About the proof of THM 3

(F) $X(\lambda)\mathcal{B}(\lambda) + Y(\lambda)\mathcal{D}(\lambda) = G(\lambda), \ \lambda \in \mathbb{C}.$

WLOG: G, X, Y are one row matrix functions. Take inverse Fourier transforms in (F):

$$x(t) + y(t) + \int_0^\omega x(s)b(t-s)\,ds + \\ + \int_{-\omega}^0 y(s)d(t-s)\,ds = g(t), \ -\omega \le t \le \omega.$$

Let S be the corresponding integral operator on $L^{1\times n}[-\omega,\omega]$. The Banach adjoint of S is the operator R on $L^n_\infty[-\omega,\omega]$ defined by

$$\big(Rf\big)(t) = \begin{cases} f(t) + \int_{-\omega}^{\omega} b(r-t)f(r)\,dr, \ t > 0, \\ f(t) + \int_{-\omega}^{\omega} d(r-t)f(r)\,dr, \ t < 0. \end{cases}$$

This operator R is the continuous analogue of the classical Sylvester resultant matrix.

About the proof of THM 3 – continued

Next, apply the Fredholm alternative together with the theorem from [1] describing the kernel of R in terms of the common root functions of $\mathcal{B}(\lambda)$ and $\mathcal{D}(\lambda)$. Commutativity (or rather quasi-commutativity) of $\mathcal{B}(\lambda)$ and $\mathcal{D}(\lambda)$ is essential.

Relevant Gohberg-K-Lerer papers:

[1] The continuous analogue of the resultant and related convolution operators, *IWOTA 2004 Proceedings*, to appear.

[2] Quasi-commutativity of entire matrix functions and the continuous analogue of the resultant, *Simonenko volume*, to appear.

[3] On a class of entire matrix function equations, *Lin.Alg.Appl.*, to appear.

[4] The inverse problem for Kreĭn orthogonal matrix functions, *J. Funct. Anal. Appl.*, to appear.