On the structure of semigroups of operators acting in spaces with indefinite metric

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Abstract

The first part of this paper concludes the cycle of studies in the structure of continuous one-parameter semigroups of operators originated in 2001 in the journal "Nonlinear Analysis" and continued in several other publications. In particular, the following theorem is proved:

If $\mathfrak{H}$ *is an* (*indefinite or definite*) *complex Krein space and* $\mathfrak{J}$ *is the K-semigroup of plus-operators acting in* $\mathfrak{H}$, *then any plus-operator* $\mathcal{F}(t) \in \mathfrak{J}$, *where* $t \geq 0$, *is a bistrict operator.*

This theorem permits removing several restrictions imposed on the sets of plus-operators in the preceding papers and thus reinforces the results of these papers.

In the second part of the present paper, we consider the heredity problem in discrete one-parametric semigroups. Namely, we study the problem of finding what indefinite properties the generating plus-operator of a semigroup and all its positive integer powers can have only simultaneously and what indefinite properties they have not necessarily simultaneously.

In conclusion, we consider several applications to dynamical systems with continuous and discrete time.

Key words: Krein space, linear operator, indefinite metric, continuous one-parameter semigroup.

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In recent years, interest in the semigroups mentioned in the title of this paper was created by the study of the so-called KE-problem.

The problem of Koenigs embedding (the KE-problem) was first used by G. Kenigs, P. Lévy, and J. Hadamard to solve various applied problems has a more than century-long history. The general statement of the problem is the following. Let D be a domain in the complex Banach space, $f \in \mathrm{Hol}(D)$. Does there exist a family $\{F(t)\}_{t\geq 0} \subset \mathrm{Hol}(D)$ continuously (in the topology of locally uniform convergence over D) depending on t and satisfying the conditions $F(0) = I$, $F(1) = f$, $F(s+t) = F(s) \circ F(t)$ for all $s, t \geq 0$?

If the family $\{F(t)\}_{t\geq 0}$ exists, then f is said to have the KE-property.

In recent years, new works concerning the KE-problem and its applications have appeared. So the case in which D is the unit open ball of the space $\mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)$, where $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are Hilbert spaces and f is a transformation of D generated by the (plus-)operator A according to the formula

$$K'_+ = \mathcal{F}_A(K_+) = (A_{21} + A_{22}K_+)(A_{11} + A_{12}K_+)^{-1}, \qquad (1)$$

with K_+, $K'_+ \in \mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ and $A_{ij} \in \mathfrak{L}(\mathfrak{H}_j, \mathfrak{H}_i)$ for i, $j = 1, 2$, was considered in [1–4]. (Here A is an operator on the space $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ with block matrix $\|A_{ij}\|$, where i, $j = 1, 2$.) In particular, in [3, 4], it was shown that the KE-property is inherent in a wide class of mappings with a fixed point in a sufficiently small neighborhood of zero.

In [5], similar results are generalized to the case of Banach spaces.

Thus, all these papers, in fact, deal with **sufficient** conditions, i.e., under these conditions, a given plus-operator can be included in the continuous semigroup of the class under study. The first part of the present paper deals with the inverse problem: what operators (of what indefinite classes) can be elements of the semigroups under study?

In the second part of the paper, we consider one-parametric discrete semigroups generated by plus-operators. Clearly, the generator of such a semigroup can have any indefinite properties. In the second part of the paper, we also investigate the heredity of such properties, namely, which of these properties can be lost or acquired in the course of the "flow."

In what follows, we consider the plus-operators in a Krein space (which has the canonical form $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, $\mathcal{J} = P_1 - P_2$, where P_1 and P_2 are mutually complementary ($P_1 + P_2 = I$) orthogonal projections onto $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively), and in a more general $\mathcal{J}_\nu$-space (i.e., a Banach space $\mathcal{B}$ decomposed into the topological direct sum of subspaces $\mathcal{B}_1$ and $\mathcal{B}_2$,

$$\mathcal{B} = \mathcal{B}_1 \dot{+} \mathcal{B}_2, \tag{2}$$

with the corresponding (bounded) mutually complementary projections P_1 and P_2: $P_1\mathcal{B} = \mathcal{B}_1$ and $P_2\mathcal{B} = \mathcal{B}_2$, endowed with $\mathcal{J}_\nu$-metric of the form

$$\mathcal{J}_\nu(x) = \|P_1 x\|^\nu - \|P_2 x\|^\nu, \quad x \in \mathcal{B}, \tag{3}$$

where $\nu > 0$, see, e.g., [6–12]).

Definition 1. If a family $\{\mathcal{F}(t)\}_{t\geq 0}$ of plus-operators acting in the $\mathcal{J}_\nu$-space is given with the operation of composition and a uniform operator topology and this family continuously depends on t and satisfies the conditions $\mathcal{F}(0) = I$ and $\mathcal{F}(s+t) = \mathcal{F}(s)\mathcal{F}(t)$ for all $s, t \geq 0$, then we shall say that a continuous (i.e., continuously depending on the parameter t) *K-semigroup* is given.

Proposition 1. *Let $\mathcal{B}$ be a $\mathcal{J}_\nu$-space, and let a K-semigroup $\mathfrak{J}$ consist of strict operators acting in $\mathcal{B}$. Then, for any $t \geq 0$, $\mathcal{F}_{11}(t) \equiv \mathcal{P}_1\mathcal{F}(t)\mathcal{P}_1|_{\mathcal{B}_1}$ is a bijective operator.*

Remark 1. We note that we do not use the algebraic properties of $\mathfrak{J}$ in this proof.

Proposition 2. *Suppose that, under the conditions of Proposition* 1*, $\mathcal{B}_1$ and $\mathcal{B}_2$ are complex Hilbert spaces. Then, for any $t \geq 0$, the operator $\mathcal{F}(t)$ is bistrict.*

Prior to proving this proposition, we need the following lemma.

Lemma 1 (Theorem 1.1 [14])**.** . *Let A be a strict plus-operator for some $\nu > 1$. Then the operator A is strict for any $\mu > 0$.*

In the case of a Krein space, the statements proved above can be generalized and reinforced.

Proposition 3. *Let $\mathcal{B} = \mathfrak{H}$ be an indefinite complex Krein space*

$$(\min\{\dim \mathfrak{H}_1, \dim \mathfrak{H}_2\} > 0),$$

and let $\mathfrak{J}$ be the K-semigroup of plus-operators acting in $\mathfrak{H}$. Then any plus-operator $\mathcal{F}(t) \in \mathfrak{J}$, where $t \geq 0$, is strict.

Remark 2. In this proof, we use the fact that the function $\mathcal{F}(t)$ is continuous only in the weak (rather than in the uniform) operator topology. In this case, using the Hahn–Banch–Sukhomlinov theorem, we can generalize our argument to the Banach spaces of several classes considered in [14].

Remark 3. In the case of a definite Krein space, it is already impossible to replace the uniform operator topology in the conditions of Proposition 3 by a weak or even strong topology. This fact is illustrated by the following example.

Let $\mathfrak{H} = \mathfrak{H}_1 = \mathfrak{L}^2(0, 1)$. For any $t \geq 0$, for $\mathcal{F}(t)$ we take the operator of multiplication by x^t. Clearly, we thus obtain a semigroup satisfying all the conditions of the K-semigroup except to the condition that $\mathcal{F}(t)$ is continuous in the uniform operator topology; but the continuity in the strong topology also holds. At the same time, for any $t > 0$, the point $\lambda = 0$ belongs to the limit spectrum of the operator $\mathcal{F}(t)$.

It is easy to prove the following assertion.

Proposition 4. *Any K-semigroup of plus-operators acting in an arbitrary* (*indefinite or definite*) $\mathcal{J}_\nu$*-space consists of homeomorphic operators.*

Theorem 1. *Let* $\mathfrak{H}$ *be an* (*indefinite or definite*) *complex Krein space, and let* $\mathfrak{J}$ *be the K-semigroup of plus-operators acting in* $\mathfrak{H}$*. Then any plus-operator* $\mathcal{F}(t) \in \mathfrak{J}$*, where* $t \geq 0$ *is bistrict.*

Now we consider the case of discrete one-parameter semigroups. We begin with the following statement about the structure of such semigroups of bistrict plus-operators.

Proposition 5. *Suppose that* A *a plus-operator in a Krein space. Suppose also that there exists a positive integer* n_0 *such that* A^{n_0} *is bistrict. Then* A *is a strict plus-operator.*

Proposition 6. *Suppose that* $\mathfrak{H}$ *is a Krein space and* A *is a strict plus-operator acting in* $\mathfrak{H}$*. Then all the operators* $(A^n)_{11}$*, where* $n \in \mathbb{N}$*, are either bijective or not bijective.*

We note that this statement also holds for Banach spaces of some classes studied in [18].

From Propositions 5 and 6, using Theorem 4.17 [15], we obtain the following assertion.

Theorem 2. *Under the conditions of Proposition* 5*, any operator* A^n*, where* $n \in \mathbb{N}$*, is bistrict.*

Now we consider a more general case of strict plus-operators in Krein spaces.

It is not difficult to construct a Krein space and a nonstrict plus-operator A in this space such that, for any $n \geq 2$, the plus-operator A^n is strict.

Example 1. Let $\{e_i^+\}_{i\in\mathbb{N}}$ be an orthonormal basis of the space $\mathfrak{H}_1$, and let $\mathfrak{H}_2 = \mathfrak{Lin}\{e^-\}$, where $[e^-,\ e^-] = -1$. It suffices to define a continuous operator A by the relations $Ae_1^+ = e_2^+ + e^-$, $Ae_i^+ = e_{i+1}^+$ for $i \geq 2$, and $Ae^- = 0$.

In Example 1, the plus-operator A is completely nonstrict, i.e., $A\mathfrak{p}_{++} \cap \mathfrak{p}_0 \neq \varnothing$. But modifying this example, we can easily make the operator A be semistrict, i.e., in this case, the operator takes any positive vector into a positive vector.

Example 2. Let $\mathfrak{H}^m$ be the Krein space from Example 1, $m \in \mathbb{N}$. We define a plus-operator A_m in the space $\mathfrak{H}^m$ by the following equalities: $A_m e^- = 0$, $A_m e_1^+ = e_2^+ + \left(1 - \frac{1}{m}\right) e^-$, and $A_m e_k^+ = e_{k+1}^+$, $k = 2, 3, \ldots$ We set $B = \bigoplus\limits_{m\in\mathbb{N}} A_m$ and B^n is strict for each $n \geq 2$.

But in the case of a Krein space $\mathfrak{H}$ with a finite-dimensional $\mathfrak{H}_1$, we have the following assertion.

Theorem 3. *Let* $\dim \mathfrak{H}_1 < \infty$*, and let* A *be a plus-operator. Then all of the operators* A^n*, where* $n \in \mathbb{N}$*, are either nonstrict or bistrict. Next, in the first case, either all of these operators are semistrict or all of them are not semistrict.*

Now we study the problem of whether the condition $\dim \mathfrak{H}_2 = \infty$ is essential in the construction of Example 2. We show that, for a rather wide class of plus-operators, this condition cannot be omitted in such constructions.

Proposition 7. *Suppose that $\mathfrak{H}$ is a Krein space,* $\dim \mathfrak{H}_2 < \infty$*, A is a plus-operator,* $\operatorname{Ker} A = \{0\}$*, $A\big|_{\mathfrak{H}_1}$ is a homeomorphism, and $A\mathfrak{H}$ is a nondegenerate lineal. Then A is a strict plus-operator.*

Remark 4. It is easy to show that if any of the three condition imposed on the plus-operator in Proposition 7 is removed, then its assertion does not hold any more.

Theorem 4. *Suppose that $\mathfrak{H}$ is a Krein space,* $\dim \mathfrak{H}_2 < \infty$*, A is a plus-operator,* $\operatorname{Ker} A = \{0\}$*, and $A\mathfrak{H}$ is a nondegenerate lineal. Then either all of the plus-operators A^n, where $n \in \mathbb{N}$, are strict or all of them are nonstrict.*

Remark 5. Any plus-operator A in Theorem 4 is semistrict. Indeed, as was shown in the proof of Proposition 7, the lineal $A\mathfrak{H}$ is positive. Therefore, since $\operatorname{Ker} A = \{0\}$, we obtain $A\mathfrak{p}_{++} \subset A(\mathfrak{H}\backslash\{0\}) \subset \mathfrak{p}_{++}$.

We note that the results of the second part of our paper naturally continue and develop the results of [20].

In conclusion, we consider several applications to dynamical systems.

In [21–26], the exponential dichotomy conditions compatible with the signature for dynamical systems of the form

$$\dot{x}(t) = A(t)x(t), \qquad x(0) = x_0, \tag{4}$$

were studied in the Hilbert space $\mathfrak{H}$ under the condition that their trajectories $x(t)$ are described by the evolution operator

$u(t,s)$, $s \le t$, $t,s \in \mathbb{R}^+$. In the general case, $u(t,s)$ satisfies the "transition conditions"

$$u(t,s) = u(t,\tau) \cdot u(\tau,s), \qquad 0 \le s \le \tau \le t.$$

In the special case of autonomous system (4) ($A(t) = A$, where A is a constant operator), the evolution operator u depends on a single parameter and determines the one-parameter semigroup

$$u(t+\tau) = u(t) \cdot u(\tau), \quad t,\tau \in \mathbb{R}^+, \qquad u(0) = I.$$

On the other hand, it follows from Theorem 1 that if all the elements of a continuous one-parameter semigroup are plus-operators, then they are bistrict plus-operators.

Therefore, the additional operator restrictions ensuring the nonemptiness of the operator sets under study, which were imposed in the preceding papers, are automatically satisfied in the case of autonomous systems.

The discrete-time dynamical systems of the form

$$y_{n+1} = A_n y_n, \qquad y_n \in \mathfrak{H}, \quad n = 0,1,2,\dots,$$

where A_n are linear continuous operators in $\mathfrak{H}$ studied in [27]. Moreover, on the one hand, [27] presents an analog of the dichotomy conditions studied in [21–26]; on the other hand, [27] presents an example of an autonomous ($A_n = A$, where A is a constant operator) nondichotomous system. The last property dramatically illustrates that the relationship between the elements of a discrete one-parameter semigroup is significantly less than that between the elements of a continuous semigroup. This fact was studied in detail in the second part of the present paper.

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