On non-canonical extensions with finitely many negative squares

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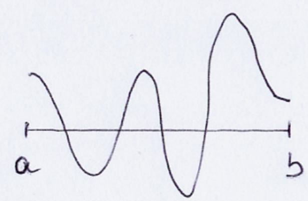
joint work with Jussi Behrndt and Carsten Trunk

Examples:

- $\ell(y) = \frac{1}{r}(-py')' + qy \quad$ on (a,b)

  a, b regular endpoints

  r can change sign



- $\ell(y)(x) = \operatorname{sign} x \left(-y''(x) + q(x)y(x)\right) \quad$ on $\mathbb{R}$

  additional condition on q . . .

$\rightsquigarrow$ self-adjoint realizations in the Kreinspace $L^2_r(a,b)$

have finitely many negative squares!

$(\mathcal{K}, [\cdot,\cdot])$ ... separable Krein space

$S \subset S^+$ .... symmetric operator, densely defined

Definition: S has κ negative squares, $\kappa \in \mathbb{N}_0$ $:\Longleftrightarrow$

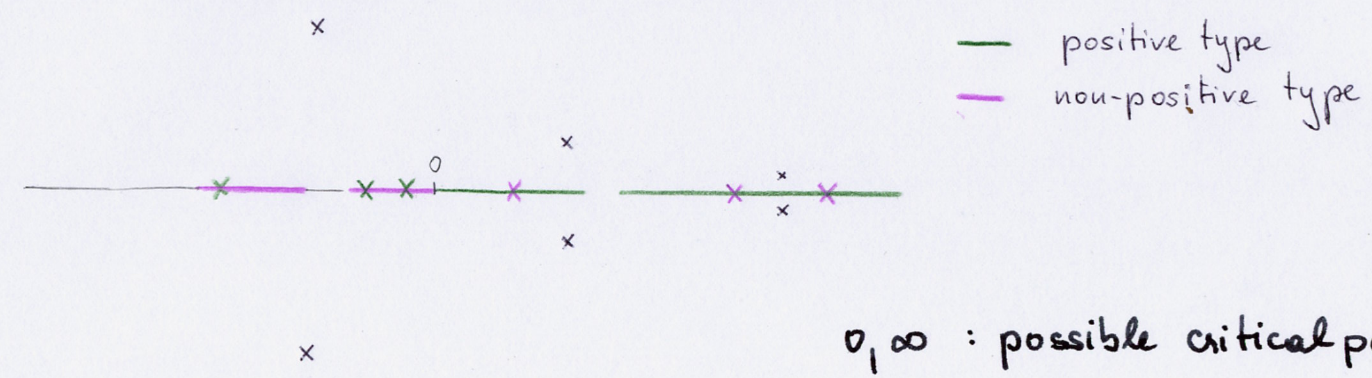
$\langle x,y \rangle := [Sx,y] \qquad x,y \in \operatorname{dom}(S)$

has κ negative squares.

$\big(\exists$ κ-dimensional subspace $\mathcal{M}$: $\langle v,v \rangle < 0 \quad \forall v \in \mathcal{M}\setminus\{0\}$
κ maximal $\big)$

Remarks: A self-adjoint, finitely many negative squares

- if $\rho(A) \neq \emptyset$ $\rightarrow$ A definitizable
- $\sigma(A)$:



$0, \infty$: possible critical points

$\tau : \mathfrak{D} \to \mathbb{C}^{n\times n}$ meromorphic in $\mathbb{C}\setminus\mathbb{R}$, symmetric $\tau(\bar\lambda) = \tau(\lambda)^*$

<u>Representation</u> : $\tau(\lambda) = \tau(\bar\lambda_0) + (\lambda - \bar\lambda_0)\Gamma^+\left(I + (\lambda-\lambda_0)(A-\lambda)^{-1}\right)\Gamma$

A… self-adjoint in $(\mathcal{K}, [\cdot,\cdot])$

$\lambda_0 \in \rho(A)$

$\Gamma \in \mathcal{L}(\mathbb{C}^n, \mathcal{K})$

$\mathcal{K}$	τ
Hilbert space	$\tau \in \mathcal{N}_0^{n\times n}$ Nevanlinna function
Pontryagin space	$\tau \in \mathcal{N}_\kappa^{n\times n}$ Generalized Nevanlinna fct
Kreinspace	…
A definitizable	τ definitizable
A κ negative squares	$\tau \in \mathcal{D}_\kappa^{n\times n}$ (Definition!)

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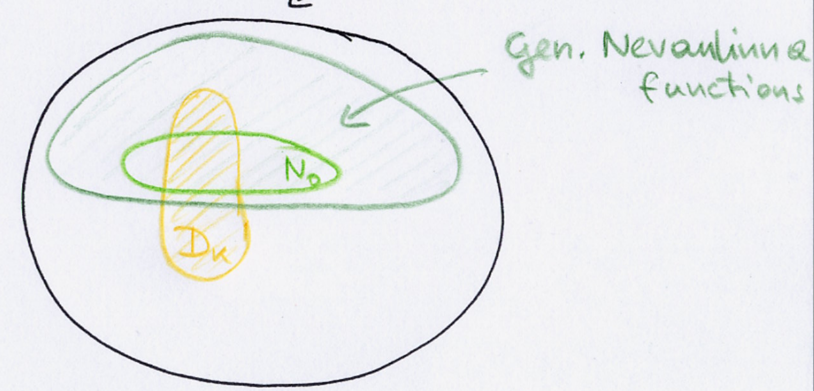
definitizable fcts.



Theorem [Jonas; Behrndt, Trunk; L]

$$\tau \in \mathcal{D}_\kappa^{n\times n} \iff \frac{\lambda}{(\lambda-\lambda_0)(\lambda-\bar\lambda_0)}\tau(\lambda) = \underbrace{Q(\lambda)}_{\in\, \mathcal{N}_\kappa^{n\times n}} + \underbrace{G(\lambda)}_{\text{rational}}$$

Q... holomorphic in $\lambda_0, \bar\lambda_0$
G... holomorphic in $\overline{\mathbb{C}} \setminus \{\lambda_0, \bar\lambda_0\}$
$\lambda_0 \in \mathrm{hol}(\tau)$

Ex: $\tau_1(\lambda) := \begin{cases} i & \operatorname{Im}\lambda > 0 \\ -i & \operatorname{Im}\lambda < 0 \end{cases}$ $\quad \tau_1 \in N_0 \setminus \mathcal{D}_\kappa$

$\tau_2(\lambda) := \lambda \cdot \tau_1(\lambda)$ $\quad \tau_2 \in \mathcal{D}_0 \setminus \mathcal{N}_\kappa$

$S \subset S^+$ … sym. operator in Kreinspace $(\mathcal{K}, [\cdot,\cdot])$; defect $(1,1)$; φ_λ … defect element

$A_0 = A_0^+$ … self-adjoint extension in $\mathcal{K}$; with κ negative squares

$m(\lambda)$ … corresponding Q-function $\to m \in \mathcal{D}_\kappa^{1\times 1}$

Krein's formula: $P_\mathcal{K} (\tilde{A}-\lambda)^{-1}|_\mathcal{K} = (A_0 - \lambda)^{-1} - \frac{1}{m(\lambda)+\tau(\lambda)} [\cdot, \varphi_{\bar{\lambda}}] \varphi_\lambda \quad (*)$

$S \subset S^+$ ... sym. operator in Kreinspace $(\mathcal{K}, [\cdot,\cdot])$; defect $(1,1)$; φ_λ ... defect element
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Krein's formula: $P_\kappa (\tilde{A}-\lambda)^{-1}|_\kappa = (A_0-\lambda)^{-1} - \frac{1}{m(\lambda)+\tau(\lambda)} [\cdot, \varphi_{\bar{\lambda}}] \varphi_\lambda \quad (*)$

Theorem:

① $(*)$ establishes a "1-1" correspondence between
- $\mathcal{K}$-minimal s.a. extensions $\tilde{A}$ with finitely many negative squares and
- parameters $\tau \in \bigcup_{\tilde{\kappa}\geq 0} \mathcal{D}_{\tilde{\kappa}}^{1\times 1} \cup \{\infty\}$

② $\tilde{A}$ has $\kappa + \kappa_\tau + \Delta_0 + \Delta_\infty$ negative squares, where $\tau \in \mathcal{D}_{\kappa_\tau}^{1\times 1}$ and

$$\Delta_0 = \begin{cases} -1 & \text{if } \exists \lim_{\lambda \hat{\to} 0} m(\lambda); \lim_{\lambda \hat{\to} 0} \tau(\lambda) \text{ and } \lim_{\lambda \hat{\to} 0} m(\lambda)+\tau(\lambda) \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Delta_\infty = \begin{cases} 0 & \text{if } \exists \lim_{\lambda \hat{\to} \infty} m(\lambda); \lim_{\lambda \hat{\to} \infty} \tau(\lambda) \text{ and } \lim_{\lambda \hat{\to} \infty} m(\lambda)+\tau(\lambda) \geq 0 \\ 1 & \text{otherwise} \end{cases}$$

canonical extensions: [Jonas, Langer 95, 06] $A_0 \geq 0$, $\kappa_\tau = 0$
[Derkach 95], [Behrndt, Trunk 07]

with exit: [Derkach, 95, 98] $\mathcal{K} = \Pi_\kappa$, finitedimensional exitspace; arbitrary defect

Proof:

1. step: „1-1" correspondence by standard methods

in particular: $\tilde{A}$ minimal representing relation for $\tilde{M}(\lambda) := -\begin{pmatrix} m(\lambda) & -1 \\ -1 & -\frac{1}{\tau(\lambda)} \end{pmatrix}^{-1}$

2. step: $\tilde{A}$ has $\tilde{\kappa}$ neg. squares $\Longleftrightarrow \tilde{M} \in \mathcal{D}_{\tilde{\kappa}}^{2\times 2} \Longleftrightarrow \frac{\lambda}{(\lambda-\lambda_0)(\lambda-\bar{\lambda}_0)}\tilde{M}(\lambda) = \underset{\mathcal{N}_{\tilde{\kappa}}^{2\times 2}}{\underset{\Uparrow}{\tilde{Q}(\lambda)}} + \tilde{G}(\lambda)$

choose: $\lambda_0 \in \mathbb{C}^+ \cap hol(m) \cap hol(-\frac{1}{m}) \cap hol(\tau) \cap hol(-\frac{1}{\tau}) \cap hol(-\frac{1}{m+\tau})$

$m \in \mathcal{D}_{\kappa_m} \rightarrow \frac{\lambda}{(\lambda-\lambda_0)(\lambda-\bar{\lambda}_0)} m(\lambda) = \underset{\mathcal{N}_{\kappa_m}}{\underset{\Uparrow}{q(\lambda)}} + \underset{\text{poles in } \lambda_0, \bar{\lambda}_0}{\underset{\nwarrow}{g(\lambda)}} \longrightarrow q_m(\lambda) := \frac{\lambda}{(\lambda-\lambda_0)(\lambda-\bar{\lambda}_0)} m(\lambda) \in \mathcal{N}_{\kappa_m+1}$

$\tau \in \mathcal{D}_{\kappa_\tau} \rightarrow \quad \ldots \quad \longrightarrow q_\tau(\lambda) := \frac{\lambda}{(\lambda-\lambda_0)(\lambda-\bar{\lambda}_0)} \tau(\lambda) \in \mathcal{N}_{\kappa_\tau+1}$

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choose: $\lambda_0 \in \mathbb{C}^+ \cap \mathrm{hol}(m) \cap \mathrm{hol}(-\frac{1}{m}) \cap \mathrm{hol}(\tau) \cap \mathrm{hol}(-\frac{1}{\tau}) \cap \mathrm{hol}(-\frac{1}{m+\tau})$

$m \in \mathcal{D}_{\kappa_m} \to \frac{\lambda}{(\lambda-\lambda_0)(\lambda-\bar{\lambda}_0)} m(\lambda) = \underset{\in\,\mathcal{N}_{\kappa_m}}{q(\lambda)} + \underset{\text{poles in } \lambda_0, \bar{\lambda}_0}{g(\lambda)} \longrightarrow q_m(\lambda) := \frac{\lambda}{(\lambda-\lambda_0)(\lambda-\bar{\lambda}_0)} m(\lambda) \in \mathcal{N}_{\kappa_m+1}$

$\tau \in \mathcal{D}_{\kappa_\tau} \to \quad \dots \quad \longrightarrow q_\tau(\lambda) := \frac{\lambda}{(\lambda-\lambda_0)(\lambda-\bar{\lambda}_0)} \tau(\lambda) \in \mathcal{N}_{\kappa_\tau+1}$

$\Longrightarrow \tilde{Q} \in \mathcal{N}_{\tilde{\kappa}}^{2\times 2} \Longleftrightarrow Q \in \mathcal{N}_{\tilde{\kappa}+2}^{2\times 2}$ where $Q = -(\tilde{Q}+\tilde{G})^{-1}$

$$Q(\lambda) = \underbrace{\begin{pmatrix} \lambda-\bar{\lambda}_0 & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} \frac{\lambda-\lambda_0}{\lambda} & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} q_m(\lambda) & -1 \\ -1 & -\frac{1}{q_\tau(\lambda)} \end{pmatrix}}_{\in\,\mathcal{N}_{\kappa_m+\kappa_\tau+2}} \begin{pmatrix} \frac{\lambda-\bar{\lambda}_0}{\lambda} & 0 \\ 0 & 1 \end{pmatrix}}_{\in\,\mathcal{N}_{\kappa_m+\kappa_\tau+2+\Delta_0}} \begin{pmatrix} \lambda-\lambda_0 & 0 \\ 0 & 1 \end{pmatrix}}_{\in\,\mathcal{N}_{\kappa_m+\kappa_\tau+2+\Delta_0+\Delta_\infty}}$$

$\tilde{\kappa} = \kappa_m + \kappa_\tau + \Delta_0 + \Delta_\infty$

$$\begin{pmatrix} \frac{\lambda-\lambda_0}{\lambda} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q_m & -1 \\ -1 & -\frac{1}{q_m} \end{pmatrix}\begin{pmatrix} \frac{\lambda-\bar\lambda_0}{\lambda} & 0 \\ 0 & 1 \end{pmatrix} \qquad \Delta_0 = \,?$$

recall: $n=1$, $q_0 \in \mathcal{N}_\kappa^{1\times 1} \longrightarrow q_1(\lambda) := \frac{\lambda-\delta}{\lambda-\bar\gamma}\, q_0(\lambda)\, \frac{\lambda-\bar\delta}{\lambda-\gamma} \in \mathcal{N}_{\kappa+\Delta}$

$$\Delta = \begin{cases} -1 & \text{if } \delta \text{ gen. pole not of positive type of } q_0,\ \gamma \text{ gen. zero — " —} \\ 1 & \text{if } \delta \text{ NOT gen. pole n. p. type of } q_0,\ \gamma \text{ NOT gen. zero — " —} \\ 0 & \text{otherwise} \end{cases}$$

here: $Q_1(\lambda) = \left(I - P^* + \frac{\lambda-\delta}{\lambda-\bar\gamma} P\right) Q_0(\lambda) \left(I - P + \frac{\lambda-\bar\delta}{\lambda-\gamma} P\right)$ $\qquad P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (\cdot, \vec e_1)\vec e_1 \qquad \delta = \lambda_0 \quad \gamma = 0$

Definition: $\alpha \in \mathbb{C}$ is called generalized pole of $Q \in \mathcal{N}_\kappa^{n\times n} \iff \exists$ holomorphic $\vec\eta : \mathcal{U}_\alpha \cap \mathbb{C}^+ \to \mathbb{C}^n$

- $\lim\limits_{\lambda \hat\to \alpha} \vec\eta(\lambda) = \vec 0$
- $\lim\limits_{\lambda \hat\to \alpha} Q(\lambda)\vec\eta(\lambda) =: \vec\eta_0 \neq \vec 0$ pole vector
- $\lim\limits_{\lambda,\omega \hat\to \alpha} \left(\frac{Q(\lambda) - Q(\bar\omega)}{\lambda - \bar\omega} \vec\eta(\lambda), \vec\eta(\omega)\right) = \gamma_0$ type of $\vec\eta_0$

$\rho \in \mathbb{C}$ is called generalized zero of $Q \in \mathcal{N}_\kappa^{n\times n} \iff \rho$ gen. pole of $-Q^{-1}$

$\iff \exists$ holomorphic $\vec\xi : \mathcal{U}_\rho \cap \mathbb{C}^+ \to \mathbb{C}^n$

- $\lim\limits_{\lambda \hat\to \rho} \vec\xi(\lambda) = \vec\xi_0 \neq \vec 0$ root vector
- $\lim\limits_{\lambda \hat\to \rho} Q(\lambda)\vec\xi(\lambda) = \vec 0$
- $\lim\limits_{\lambda,\omega \hat\to \rho} \left(\frac{Q(\lambda) - Q(\bar\omega)}{\lambda - \bar\omega} \vec\xi(\lambda), \vec\xi(\omega)\right) = \gamma_0$ type of $\vec\xi_0$

Lemma 1 [Lu. 02, 03]

$Q_0 \in \mathcal{N}_{\kappa_0}^{n\times n}$; $\quad \gamma, \delta \in \mathbb{C}$, $\gamma \neq \bar{\delta}$; $\quad \vec{\psi}, \vec{\varphi} \in \mathbb{C}$, $\vec{\psi} \not\perp \vec{\varphi}$; $\quad$ projection: $P = \dfrac{(\cdot, \vec{\varphi})}{(\vec{\psi}, \vec{\varphi})} \vec{\psi}$

$$Q_1(\lambda) := \left(I - P^* + \frac{\lambda - \delta}{\lambda - \bar{\gamma}} P^*\right) Q_0(\lambda) \left(I - P + \frac{\lambda - \bar{\delta}}{\lambda - \gamma} P\right) \in \mathcal{N}_{\kappa_0 + \Delta}^{n\times n}$$

1) δ … gen. pole of Q $\quad \vec{\varphi}$ … corr. non-pos. ~~root~~ pole vector
γ … gen. zero of Q $\quad \vec{\psi}$ … corr. non-pos. root vector
$\rightarrow$ $\Delta = -1$

2) δ … gen. pole of Q $\quad \vec{\varphi}$ … corr. non-pos. pole vector
γ … neither gen. zero nor pole $\quad \vec{\psi}$ … arbitrary
$\rightarrow$ $\Delta = 0$

⋮

Lemma 2 γ not gen. zero of Q_0 $\Longrightarrow$ γ gen. pole of Q_1 with non-pos. pole vector $\vec{\varphi}$

Lemma 3 [Behrndt, L07]

$Q_0 = \begin{pmatrix} q_m & -1 \\ -1 & -\frac{1}{q_\tau} \end{pmatrix}$ $\Longrightarrow$ $w_0 \in \mathbb{C}$ is gen. zero of Q_0 $\Longleftrightarrow$

w_0 gen. zero of $q_m + q_\tau$ $\qquad \vec{\xi}_0 = \begin{pmatrix} 1 \\ -q_\tau(w_0) \end{pmatrix}$

or

w_0 gen. pole of q_m and q_τ $\qquad \vec{\xi}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Delta_0 = ?$

$$Q_1(\lambda) = \left(I - P^* + \frac{\lambda - \delta}{\lambda - \bar{\gamma}} P^*\right) Q_0(\lambda) \left(I - P + \frac{\lambda - \bar{\delta}}{\lambda - \bar{\gamma}} P\right) = \begin{pmatrix} \frac{\lambda - \lambda_0}{\lambda} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_m(\lambda) & -1 \\ -1 & -\frac{1}{q_\tau(\lambda)} \end{pmatrix} \begin{pmatrix} \frac{\lambda - \bar{\lambda}_0}{\lambda} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{m(\lambda)}{\lambda} & -\frac{\lambda - \lambda_0}{\lambda} \\ -\frac{\lambda - \bar{\lambda}_0}{\lambda} & -\frac{(\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)}{\lambda} \cdot \frac{1}{\tau(\lambda)} \end{pmatrix}$$

$$Q_0(\lambda) = \begin{pmatrix} \frac{\lambda}{(\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)} m(\lambda) & -1 \\ -1 & -\frac{(\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)}{\lambda} \cdot \frac{1}{\tau(\lambda)} \end{pmatrix}$$

$\delta_0 = \lambda_0 \ldots$ gen. pole of Q_0 with non-pos. pole vector $\binom{1}{0}$

case A: Assume $\gamma = 0$ is a gen. zero of Q_0 with non-pos. root vector $\binom{1}{0}$ $\Rightarrow \Delta_0 = -1$

$\Updownarrow$

0 is gen. zero of both q_m and q_τ and $\lim\limits_{\lambda \to 0} \frac{q_m(\lambda) + q_\tau(\lambda)}{\lambda} \leq 0$ (type!)

$\Updownarrow \ldots$

$\exists \lim\limits_{\lambda \to 0} m(\lambda)$ and $\lim\limits_{\lambda \to 0} \tau(\lambda)$ and $\lim\limits_{\lambda \to 0} m(\lambda) + \tau(\lambda) \leq 0$

$\Delta_0 = ?$

$$Q_1(\lambda) = \left(I-P^* + \frac{\lambda-\delta}{\lambda-\bar{\gamma}}P^*\right) Q_0(\lambda) \left(I-P+\frac{\lambda-\bar{\delta}}{\lambda-\bar{\gamma}}P\right) = \begin{pmatrix} \frac{\lambda-\lambda_0}{\lambda} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_m(\lambda) & -1 \\ -1 & -\frac{1}{q_\tau(\lambda)} \end{pmatrix} \begin{pmatrix} \frac{\lambda-\bar{\lambda}_0}{\lambda} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{m(\lambda)}{\lambda} & -\frac{\lambda-\lambda_0}{\lambda} \\ -\frac{\lambda-\bar{\lambda}_0}{\lambda} & -\frac{(\lambda-\lambda_0)(\lambda-\bar{\lambda}_0)}{\lambda}\cdot\frac{1}{\tau(\lambda)} \end{pmatrix}$$

$$Q_0(\lambda) = \begin{pmatrix} \frac{\lambda}{(\lambda-\lambda_0)(\lambda-\bar{\lambda}_0)} m(\lambda) & -1 \\ -1 & -\frac{(\lambda-\lambda_0)(\lambda-\bar{\lambda}_0)}{\lambda}\cdot\frac{1}{\tau(\lambda)} \end{pmatrix}$$

$\delta = \lambda_0$ … gen. pole of Q_0 with non-pos. pole vector $\binom{1}{0}$

case A: Assume $\gamma = 0$ is a gen. zero of Q_0 with non-pos. rootvector $\binom{1}{0}$ $\Rightarrow \Delta_0 = -1$

$\Updownarrow$

0 is gen. zero of both q_m and q_τ and $\lim_{\lambda\to 0} \frac{q_m(\lambda)+q_\tau(\lambda)}{\lambda} \leq 0$ (type!)

$\Updownarrow$ …

$\exists \lim_{\lambda\to 0} m(\lambda)$ and $\lim_{\lambda\to 0}\tau(\lambda)$ and $\lim_{\lambda\to 0} m(\lambda)+\tau(\lambda) \leq 0$

case B: Assume $\gamma = 0$ is NOT a gen. zero of Q_0 $\rightsquigarrow$ consider $Q_0 = \begin{pmatrix} \frac{\lambda}{\lambda-\lambda_0} & 0 \\ 0 & 1 \end{pmatrix} Q_1 \begin{pmatrix} \frac{\lambda}{\lambda-\bar{\lambda}_0} & 0 \\ 0 & 1 \end{pmatrix}$

0 … gen. pole of Q_1 with non-pos. rootvector $\binom{1}{0}$

λ_0 … NO pole of Q_1, but zero with rootvector $\binom{0}{1}$ $\rightsquigarrow \tilde{Q}_1 := Q_1 + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\Rightarrow \Delta_0 = 0$

cases C, D:

Assume 0 is a gen. zero of Q_0 $\to$ with pos. rootvector $\binom{1}{0}$

$\to$ $\binom{1}{0}$ is not rootvector … $\Rightarrow \Delta_0 = 0$

$\Delta_\infty = ?$ …