WIAS

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Wigner's R-matrix and Weyl functions

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1 Basic facts on scattering system

Pair of self-adjoint operators $\{L, L_0\}$ on some separable Hilbert space $\mathfrak{L}$ such that the wave operators

$$W_\pm(L, L_0) = s - \lim_{t\to\pm\infty} e^{itL}e^{-itL_0}P^{ac}(L_0)$$

exists where $P^{ac}(L_0)$ is the projection onto the absolutely continuous subspace $\mathfrak{L}^{ac}(L_0)$ of L_0. One has

$$\mathrm{ran}(W_\pm(L, L_0)) \subseteq \mathfrak{L}^{ac}(L_0).$$

We say the scattering system is complete if

$$\mathrm{ran}(W_\pm(L, L_0)) = \mathfrak{L}^{ac}(L).$$

It is known that completeness $\Longleftrightarrow$ existence of $W_\pm(L_0, L)$.

2 Existence of the wave operators

Let $V = V^*$ be a self-adjoint trace class operator. If

$$L = L_0 + V,$$

then $\{L, L_0\}$ performs a complete scattering system.

If

$$(L - z)^{-p} - (L_0 - z)^{-p} \in \mathcal{B}_1(\mathfrak{L}), \quad p \in \mathbb{N},$$

for some $z \in \mathbb{C} \setminus \mathbb{R}$, then $\{L, L_0\}$ is a complete scattering system. In particular, if

$$(L - z)^{-1} - (L_0 - z)^{-1} \in \mathcal{B}_1(\mathfrak{L}).$$

3 Scattering operator

The scattering operator $\boldsymbol{S} : \mathfrak{L}^{ac}(\boldsymbol{L}_0) \longrightarrow \mathfrak{L}^{ac}(\boldsymbol{L}_0)$

$$\boldsymbol{S}(\boldsymbol{L}, \boldsymbol{L}_0) := \boldsymbol{W}_+(\boldsymbol{L}, \boldsymbol{L}_0)^* \boldsymbol{W}_-(\boldsymbol{L}, \boldsymbol{L}_0).$$

Obviously, one has

$$e^{-it\boldsymbol{L}_0} \boldsymbol{S}(\boldsymbol{L}, \boldsymbol{L}_0) = \boldsymbol{S}(\boldsymbol{L}, \boldsymbol{L}_0) e^{-it\boldsymbol{L}_0}, \quad t \in \mathbb{R},$$

which is equivalent to

$$\boldsymbol{E}_0(\boldsymbol{\Delta}) \boldsymbol{S}(\boldsymbol{L}, \boldsymbol{L}_0) = \boldsymbol{S}(\boldsymbol{L}, \boldsymbol{L}_0) \boldsymbol{E}_0(\boldsymbol{\Delta}), \quad \boldsymbol{\Delta} \in \mathfrak{B}(\mathbb{R}).$$

If $\{\boldsymbol{L}, \boldsymbol{L}_0\}$ is a complete scattering system, then $\boldsymbol{S}(\boldsymbol{L}, \boldsymbol{L}_0)$ is unitary on $\mathfrak{L}^{ac}(\boldsymbol{L}_0)$, that is,

$$\boldsymbol{S}(\boldsymbol{L}, \boldsymbol{L}_0)^* \boldsymbol{S}(\boldsymbol{L}, \boldsymbol{L}_0) = \boldsymbol{S}(\boldsymbol{L}, \boldsymbol{L}_0) \boldsymbol{S}(\boldsymbol{L}, \boldsymbol{L}_0)^* = \boldsymbol{I}_{\mathfrak{L}^{ac}(\boldsymbol{L}_0)}.$$

4 Scattering matrix

There is direct integral representation of $\mathfrak{L}^{ac}(L_0)$,

$$\mathfrak{L}^{ac}(L_0) \cong \int^{\oplus} \mathfrak{Q}_\lambda d\mu(\lambda),$$

where $\{\mathfrak{Q}_\lambda\}_{\lambda\in\mathbb{R}}$ is family of Hilbert spaces and $\mu(\cdot)$ is a Borel measure on $\mathbb{R}$ which is absolutely continuous with respect to the Lebesgue measure $d\lambda$ on $\mathbb{R}$, such that

$$L_0^{ac} \cong \lambda$$

Such a representation is called a spectral representation of L_0^{ac}.

Since $S(L, L_0)$ commutes with L_0^{ac}, there is a measurable family of operators $\{S(\lambda)\}_{\lambda\in\mathbb{R}}$, $S(\lambda) : \mathfrak{Q}_\lambda \longrightarrow \mathfrak{Q}_\lambda$, such that

$$S(L, L_0) \cong S(\lambda)$$

If $S(L, L_0)$ is unitary, then $S(\lambda)$ is unitary for a.e. λ with respect to μ. $\{S(\lambda)\}_{\lambda\in\mathbb{R}}$ is called the scattering matrix of the scattering system $\{L, L_0\}$.

5 Example

$$Lf = -\frac{1}{2}\frac{d}{dx}\frac{1}{M}\frac{d}{dx}f + Vf, \quad f \in \mathrm{dom}(L) = \{f \in W^{1,2}(\mathbb{R}) : \frac{1}{M}f \in W^{1,2}(\mathbb{R})\}.$$

where

$$M(x) := \begin{cases} m_l, & x \in (-\infty, x_l] \\ m(x), & x \in (x_l, x_r) \\ m_r, & x \in [x_r, \infty) \end{cases} \qquad V(x) := \begin{cases} v_l, & x \in (-\infty, x_l] \\ v(x), & x \in (x_l, x_r) \\ v_r, & x \in [x_r, \infty). \end{cases}$$

$$L_0 := -\frac{1}{2m_l}\frac{d^2}{dx^2} + v_l \oplus -\frac{1}{2}\frac{d}{dx}\frac{1}{m}\frac{d}{dx} + v(x) \oplus -\frac{1}{2m_r}\frac{d^2}{dx^2} + v_r \qquad \text{Dirichlet b. c.}$$

$$L^2(\mathbb{R}) = L^2((-\infty), x_l) \oplus L^2((x_l, x_r)) \oplus L^2((x_r, \infty).$$

$\{L, L_0\}$ performs a complete scattering system

6 Eisenbud-Wigner representation

Let $\{S(\lambda)\}_{\lambda\in\mathbb{R}}$ be the scattering matrix in the spectral representation

$$\int^{\oplus} \mathfrak{Q}_\lambda d\mu(\lambda) \simeq L^2((v_r, v_l), \mathbb{C}) \oplus L^2((v_l, \infty), \mathbb{C}^2), \qquad v_l > v_r.$$

Wigner's $\boldsymbol{R}$-matrix:

$$R(\lambda) := i(I_{\mathfrak{Q}_\lambda} - S(\lambda))(I_{\mathfrak{Q}_\lambda} + S(\lambda))^{-1} \quad \Longrightarrow \quad S(\lambda) := \frac{iI_{\mathfrak{Q}_\lambda} - R(\lambda)}{iI_{\mathfrak{Q}_\lambda} + R(\lambda)}$$

$$R(\lambda) = \sum_{k=1}^{\infty} (\lambda_k - \lambda)^{-1} \left(\cdot, \begin{pmatrix} \sqrt[4]{\frac{\lambda - v_l}{2m_l}} \psi_k(x_l, \lambda) \\ \sqrt[4]{\frac{\lambda - v_r}{2m_r}} \psi_k(x_r, \lambda) \end{pmatrix} \right) \begin{pmatrix} \sqrt[4]{\frac{\lambda - v_l}{2m_l}} \psi_k(x_l, \lambda) \\ \sqrt[4]{\frac{\lambda - v_r}{2m_r}} \psi_k(x_r, \lambda) \end{pmatrix}, \qquad \lambda > v_l,$$

where $\{\lambda_k\}$ and ψ_k, $k = 1, 2, \ldots$, are the eigenvalues and eigenfunctions of the selfadjoint operator

$$A_1 := -\frac{1}{2}\frac{d}{dx}\frac{1}{m(x)}\frac{d}{dx} + v(x), \qquad \text{Neumann b. c.}$$

7 Boundary triplets and scattering

Let A be a closed symmetric operator on $\mathfrak{H}$ and $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet of A^*.

By $M(z)$ we denote the corresponding Weyl function.

We consider the extensions

$$A_0 := L^* \upharpoonright \ker(\Gamma_0) \quad \text{and} \quad A_\Theta := L^* \upharpoonright \Gamma^{-1}\Theta \tag{1}$$

where Θ is some self-adjoint relation on $\mathcal{H}$.

If the deficiency indices of L are finite, then any pair $\{A_\Theta, A_0\}$ performs a complete scattering system since the resolvent difference is a finite dimensional operator.

Problem: Let us consider the scattering system $\{L_\Theta, L_0\}$. Is it possible to calculate the scattering matrix $\{S(\lambda)\}_{\lambda\in\mathbb{R}}$?

8 Boundary triplets and direct integrals

Since $\mathcal{H}$ is finite dimensional the limits $M(\lambda) := M(\lambda + i0)$ for a.e. $\lambda \in \mathbb{R}$. We denote by $\Sigma^M \subseteq \mathbb{R}$ the set where the limit $M(\lambda + i0)$ exists. Further,

$$\mathcal{H}_{M(\lambda)} := \mathrm{ran}(\Im\mathrm{m}(M(\lambda)) \subseteq \mathcal{H}, \quad \lambda \in \Sigma^\tau.$$

By $Q_{M(\lambda)}$ we denote the family of orthogonal projections onto $\mathcal{H}_{M(\lambda)}$ which is measurable.

$$L^2(\mathbb{R}, d\lambda, \mathcal{H}) = \int^\oplus \mathcal{H} d\lambda.$$

In $L^2(\mathbb{R}, d\lambda, \mathcal{H})$ we introduce the projection

$$(Q_M f)(\lambda) := Q_{M(\lambda)} f(\lambda), \quad f \in L^2(\mathbb{R}, d\lambda, \mathcal{H}).$$

The subspace $Q_M L^2(\mathbb{R}, d\lambda, \mathcal{H})$ is denoted by

$$Q_M L^2(\mathbb{R}, d\lambda, \mathcal{H}) = L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)}) = \int^\oplus \mathcal{H}_{M(\lambda)} d\lambda.$$

It turns out that $L_0^{ac} \cong \lambda$ where λ is the multiplication operator in $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)})$.

9 Boundary triplets and scattering

THEOREM 1. *Let A be a densely defined closed simple symmetric operator with finite deficiency indices in the separable Hilbert space $\mathfrak{H}$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for A^* and $M(\cdot)$ be the corresponding Weyl function. Further, let $A_0 = A^* \upharpoonright \ker(\Gamma_0)$ and let $A_\Theta = A^* \upharpoonright \Gamma^{-1}\Theta$ be a self-adjoint extension of A where Θ is a self-adjoint relation in $\mathcal{H}$. Then the scattering matrix $\{S(\lambda)\}_{\lambda \in \mathbb{R}}$ of the complete scattering system $\{A_\Theta, A_0\}$ admits the representation*

$$S(\lambda) = I_{\mathcal{H}_{M(\lambda)}} + 2i\sqrt{\Im\mathrm{m}(M(\lambda))}\big(\Theta - M(\lambda)\big)^{-1}\sqrt{\Im\mathrm{m}(M(\lambda))}$$

for a.e. $\lambda \in \mathbb{R}$, where $M(\lambda) := M(\lambda + i0)$.

10 Open quantum systems and decoupled system

Let us consider two symmetric operators A and T in $\mathfrak{H}$ and $\mathfrak{K}$, respectively, with equal finite deficiency indices. Further, let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ and $\{\mathcal{H}, \Upsilon_0, \Upsilon_1\}$ boundary triplets with Weyl functions $M(\lambda)$ and $\tau(\lambda)$, respectively. Then $\{\widetilde{\mathcal{H}}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$,

$$\widetilde{\mathcal{H}} := \begin{pmatrix} \mathcal{H} \\ \mathcal{H} \end{pmatrix}, \qquad \widetilde{\Gamma}_0 := \begin{pmatrix} \Gamma_0 \\ \Upsilon_0 \end{pmatrix}, \qquad \widetilde{\Gamma}_1 := \begin{pmatrix} \Gamma_1 \\ \Upsilon_1 \end{pmatrix}$$

performs a boundary triplet for $A^* \oplus T^*$ with Weyl function

$$\widetilde{M}(\lambda) := \begin{pmatrix} M(\lambda) & 0 \\ 0 & \tau(\lambda) \end{pmatrix}$$

The systems $\{\mathfrak{H}, A\}$ and $\{\mathfrak{K}, T\}$ are called open system, $\{\mathfrak{H}, A\}$ is called the inner system, $\{\mathfrak{K}, T\}$ is called the outer system. The observer is in the inner system.

The system $\{\mathfrak{L}, A_0 \oplus T_0\}$, $A_0 := A^* \upharpoonright \ker(\Gamma_0)$, $T_0 := T^* \upharpoonright \ker(\Upsilon_0)$, is called the decoupled system.

11 Open quantum system and coupled system

THEOREM 2 (Derkach, Hassi, M. de Snoo, 2000)**.** *Let A and T be densely defined closed symmetric operators in the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ which equal deficiency indices. Then the following holds:*

(i) *The closed extension $L := A^* \oplus T^* \restriction \widetilde{\Gamma}^{-1}\widetilde{\Theta}$ corresponding to the relation*

$$\widetilde{\Theta} := \left\{ \begin{pmatrix} (v,v)^\top \\ (w,-w)^\top \end{pmatrix} : v, w \in \mathcal{H} \right\}$$

is self-adjoint in the Hilbert space $\mathfrak{L} := \mathfrak{H} \oplus \mathfrak{K}$ and is given by

$$L = A^* \oplus T^* \restriction \left\{ f_1 \oplus f_2 \in \mathrm{dom}(A^* \oplus T^*) : \begin{array}{c} \Gamma_0 f_1 - \Upsilon_0 f_2 = 0 \\ \Gamma_1 f_1 + \Upsilon_1 f_2 = 0 \end{array} \right\}.$$

(ii) *The Strauss family $A_{-\tau(\lambda)} := A^* \restriction \ker(\Gamma_1 + \tau(\lambda)\Gamma_0)$, $\lambda \in \mathbb{C}_+$, satisfies*

$$(A_{-\tau(\lambda)} - \lambda)^{-1} = P_{\mathfrak{H}}(L - \lambda)^{-1} \restriction \mathfrak{H}, \qquad \lambda \in \mathbb{C}_+.$$

The system $\{\mathfrak{L}, L\}$ is called the coupled system.

12 Strauss family

Let $\tau(\cdot):\mathcal{K}\longrightarrow\mathcal{K}$ be a Nevanlinna function.

$$A_{-\tau(\lambda)} := A^*\restriction\left\{f\in\mathrm{dom}(A^*) : \Gamma_1 f = -\tau(\lambda)\Gamma_0 f\right\},\quad \lambda\in\mathbb{C}_+,$$

$\{A_{-\tau(\lambda)}\}_{\lambda\in\mathbb{C}_+}$ is called a Strauss family.

Since $\dim(\mathcal{H})<\infty$ the family admits an extension to a.e. $\lambda\in\mathbb{R}$, i.e.

$$\tau(\lambda) := \lim_{\epsilon\longrightarrow+0}\tau(\lambda+i\epsilon).$$

In general, the Strauss family consists of maximal dissipative operators. The characteristic function of $A_{-\tau(\lambda)}$ are given by

$$\Theta_{A_{-\tau(\lambda)}}(\mu) = I_{\mathcal{Q}_\lambda} + 2i\sqrt{\Im\mathrm{m}(\tau(\lambda))}\left(\tau(\lambda)^* + M(\overline{\mu})^*\right)^{-1}\sqrt{\Im\mathrm{m}(\tau(\lambda))},\quad \mu\in\mathbb{C}_-,$$

where

$$\mathcal{Q}_\lambda := \mathrm{clo}\{\mathrm{ran}(\Im\mathrm{m}\tau(\lambda))\}.$$

13 Coupling and scattering

THEOREM 3. *Let A and T be densely defined closed simple symmetric operators in $\mathfrak{H}$ and $\mathfrak{K}$, respectively, with equal finite deficiency indices such that A_0 is discrete. Then*

(i) *The wave operators*

$$W_\pm(L, L_0) = s - \lim_{t\to\pm\infty} e^{itL} e^{-itL_0} P^{ac}(L_0) = s - \lim_{t\to\pm\infty} e^{itL} e^{-itT_0} P^{ac}(T_0)$$

exist and are complete.

(ii) *The scattering matrix $\{S(\lambda)\}_{\lambda\in\mathbb{R}}$ of the scattering system $\{L, L_0\}$ admits the representation*

$$S(\lambda) = I_{\mathfrak{Q}_\lambda} - 2i\sqrt{\Im m\tau(\lambda)}\big(\tau(\lambda) + M(\lambda)\big)^{-1}\sqrt{\Im m\tau(\lambda)}$$

for a.e. $\lambda \in \mathbb{R}$, where $\tau(\lambda) = \tau(\lambda + i0)$ and $M(\lambda) = M(\lambda + i0)$.

(iii) *The scattering matrix $\{S(\lambda)\}_{\lambda\in\mathbb{R}}$ of the scattering system $\{L, L_0\}$ admits the representation*

$$S(\lambda) = \Theta_{A_{-\tau(\lambda)}}(\lambda - i0)^* \tag{2}$$

for a.e. $\lambda \in \mathbb{R}$ where $\Theta_{A_{-\tau(\lambda)}}(\cdot)$, $\lambda \in \mathbb{R}$, are the characteristic functions of the the Strauss family $\{A_{-\tau(\lambda)}\}_{\lambda\in\mathbb{R}}$.

14 R-matrix

One introduces the $\boldsymbol{R}$-matrix

$$\boldsymbol{R}(\boldsymbol{\lambda}) := \boldsymbol{i}(\boldsymbol{I}_{\mathcal{H}_{\tau(\lambda)}} - \boldsymbol{S}(\boldsymbol{\lambda}))(\boldsymbol{I}_{\mathcal{H}_{\tau(\lambda)}} + \boldsymbol{S}(\boldsymbol{\lambda}))^{-1},$$

for those $\boldsymbol{\lambda} \in \boldsymbol{\Sigma}^{\tau}$ obeying $-1 \in \varrho(\widetilde{\boldsymbol{S}}(\boldsymbol{\lambda}))$ which is a bounded operator acting in $\mathcal{H}_{\tau(\lambda)}$. Conversely, one has

$$\boldsymbol{S}(\boldsymbol{\lambda}) = \frac{\boldsymbol{i}\boldsymbol{I}_{\mathcal{H}(\tau(\lambda))} - \boldsymbol{R}(\boldsymbol{\lambda})}{\boldsymbol{i}\boldsymbol{I}_{\mathcal{H}(\tau(\lambda))} + \boldsymbol{R}(\boldsymbol{\lambda})}$$

A straightforward calculation shows that

$$\boldsymbol{R}(\boldsymbol{\lambda}) = -\sqrt{\Im\mathrm{m}(\boldsymbol{\tau}(\boldsymbol{\lambda}))}\,(\boldsymbol{M}(\boldsymbol{\lambda}) + \Re\mathrm{e}(\boldsymbol{\tau}(\boldsymbol{\lambda})))^{-1})\sqrt{\Im\mathrm{m}(\boldsymbol{\tau}(\boldsymbol{\lambda}))}$$

for $\boldsymbol{\lambda} \in \{\boldsymbol{t} \in \boldsymbol{\Sigma}^{\tau} : \Im\mathrm{m}(\boldsymbol{\tau}(\boldsymbol{t})) \neq \boldsymbol{0}\} \cap \boldsymbol{\Sigma}^{M} \cap \boldsymbol{\Sigma}^{(M+\tau)^{-1}}$ and $\ker(\boldsymbol{M}(\boldsymbol{\lambda}) + \Re\mathrm{e}(\boldsymbol{\tau}(\boldsymbol{\lambda})) = \{0\}$. If $\Re\mathrm{e}(\boldsymbol{\tau}(\boldsymbol{\lambda})) = \boldsymbol{0}$, then

$$\boldsymbol{R}(\boldsymbol{\lambda}) = -\sqrt{\Im\mathrm{m}(\boldsymbol{\tau}(\boldsymbol{\lambda}))}\boldsymbol{M}(\boldsymbol{\lambda})^{-1}\sqrt{\Im\mathrm{m}(\boldsymbol{\tau}(\boldsymbol{\lambda}))}$$

for $\boldsymbol{\lambda} \in \{\boldsymbol{t} \in \boldsymbol{\Sigma}^{\tau} : \Im\mathrm{m}(\boldsymbol{\tau}(\boldsymbol{t})) \neq \boldsymbol{0}\} \cap \boldsymbol{\Sigma}^{M} \cap \boldsymbol{\Sigma}^{(M+\tau)^{-1}}$ and $\ker(\boldsymbol{M}(\boldsymbol{\lambda})) = \{0\}$.

15 Eigenfunction representation

Let us introduce the self-adjoint extensions

$$\boldsymbol{A}_{-\Re\mathrm{e}(\boldsymbol{\tau}(\boldsymbol{\lambda}))} := \boldsymbol{A}^* \restriction \ker(\boldsymbol{\Gamma}_1 + \Re\mathrm{e}(\boldsymbol{\tau}(\boldsymbol{\lambda}))\boldsymbol{\Gamma}_0)$$

for $\boldsymbol{\lambda} \in \Sigma^{\tau}$. If $\boldsymbol{\lambda} \in \Sigma^{\tau} \cap \varrho(\boldsymbol{A}_0)$, then

$$\boldsymbol{\lambda} \in \varrho(\boldsymbol{A}_{-\Re\mathrm{e}(\boldsymbol{\tau}(\boldsymbol{\lambda}))}) \Longleftrightarrow \ker(\boldsymbol{M}(\boldsymbol{\lambda}) + \Re\mathrm{e}(\boldsymbol{\tau}(\boldsymbol{\lambda}))) = \{\boldsymbol{0}\}.$$

PROPOSITION 4. *Let* $\boldsymbol{A}$, $\{\boldsymbol{\mathcal{H}}, \boldsymbol{\Gamma}_0, \boldsymbol{\Gamma}_1\}$, $\boldsymbol{M}(\cdot)$ *and* $\boldsymbol{T}$, $\{\boldsymbol{\mathcal{H}}, \boldsymbol{\Upsilon}_0, \boldsymbol{\Upsilon}_1\}$, $\boldsymbol{\tau}(\cdot)$ *be as above and assume* $\boldsymbol{\sigma}(\boldsymbol{A}_0) = \boldsymbol{\sigma}_p(\boldsymbol{A}_0)$ *and that* $\boldsymbol{A}$ *is semibounded from below. For each* $\boldsymbol{\lambda} \in \{\boldsymbol{t} \in \Sigma^{\tau} : \Im\mathrm{m}\boldsymbol{\tau}(\boldsymbol{t})) \neq \boldsymbol{0}\} \cap \varrho(\boldsymbol{A}_0) \cap \varrho(\boldsymbol{A}_{-\boldsymbol{\tau}(\boldsymbol{\lambda})}) \cap \varrho(\boldsymbol{A}_{-\Re\mathrm{e}(\boldsymbol{\tau}(\boldsymbol{\lambda}))})$ *with* $\boldsymbol{A}_{-\Re\mathrm{e}(\boldsymbol{\tau}(\boldsymbol{\lambda}))} \leq \boldsymbol{A}_0$ *the* $\boldsymbol{R}$*-matrix admits the representation*

$$\boldsymbol{R}(\boldsymbol{\lambda}) = \sum_{k=1}^{\infty} (\boldsymbol{\lambda}_k[\boldsymbol{\lambda}] - \boldsymbol{\lambda})^{-1} \left(\cdot, \sqrt{\Im\mathrm{m}(\boldsymbol{\tau}(\boldsymbol{\lambda}))}\boldsymbol{\Gamma}_0\psi_k[\boldsymbol{\lambda}]\right) \sqrt{\Im\mathrm{m}(\boldsymbol{\tau}(\boldsymbol{\lambda}))}\boldsymbol{\Gamma}_0\psi_k[\boldsymbol{\lambda}],$$

where $\{\boldsymbol{\lambda}_k[\boldsymbol{\lambda}]\}$, $\boldsymbol{k} = \boldsymbol{1}, \boldsymbol{2}, \ldots$, *are the eigenvalues of the selfadjoint extension* $\boldsymbol{A}_{-\Re\mathrm{e}(\boldsymbol{\tau}(\boldsymbol{\lambda}))}$ *in increasing order and* $\psi_k[\boldsymbol{\lambda}]$ *are the corresponding eigenfunctions.*

16 Wigner-Eisenbud representation

COROLLARY 5 (Wigner-Eisenbud '46–'47)**.** *If in addition* $\Re e(\tau(\lambda)) = 0$ *and* $A_1 \leq A_0$, *then for each* $\lambda \in \{t \in \Sigma^\tau : \Im m(\tau(t)) \neq 0\} \cap \varrho(A_0) \cap \varrho(A_{-\tau(\lambda)}) \cap \varrho(A_1)$ *the* R*-matrix admits the representation*

$$R(\lambda) = \sum_{k=1}^{\infty} (\lambda_k - \lambda)^{-1} \big(\cdot, \sqrt{\Im m(\tau(\lambda))}\Gamma_0 \psi_k\big) \sqrt{\Im m(\tau(\lambda))}\Gamma_0 \psi_k,$$

where $\{\lambda_k\}$, $k = 1, 2, \ldots$, *are the eigenvalues of the selfadjoint extension* $A_1 := A^* \upharpoonright \ker(\Gamma_1)$ *in increasing order and* ψ_k *are the corresponding eigenfunctions.*

In particular, if $A_0 := A^* \upharpoonright \ker(\Gamma_0)$ is the Friedrichs extension, then the condition $A_{-\Re e(\tau(\lambda))} \leq A_0$ or $A_1 \leq A_0$ is always satisfied.

If the condition $A_{-\Re e(\tau(\lambda))} \leq A_0$ or $A_1 \geq A_0$ is not satisfied, then Wigner-Eisenbud representation is not true.

17 Example

17.1 Inner system

In $\mathfrak{H} := L^2((x_l, x_r))$ one defines

$$(Af)(x) := -\frac{1}{2}\frac{d}{dx}\frac{1}{m(x)}\frac{d}{dx}f(x) + v(x)f(x),$$

$$\mathrm{dom}(A) := \left\{ f \in \mathfrak{H} : \begin{array}{c} f, \frac{1}{m}f' \in W^{1,2}((x_l, x_r)) \\ f(x_l) = f(x_r) = 0 \\ \left(\frac{1}{m}f'\right)(x_l) = \left(\frac{1}{m}f'\right)(x_r) = 0 \end{array} \right\}.$$

where $m > 0$ and $m + \frac{1}{m} \in L^\infty((x_l, x_r))$, $v \in L^\infty((x_l, x_r))$.

$$\Gamma_0 f := \begin{pmatrix} f(x_l) \\ f(x_r) \end{pmatrix} \quad \text{and} \quad \Gamma_1 f := \frac{1}{2}\begin{pmatrix} \left(\frac{1}{m}f'\right)(x_l) \\ -\left(\frac{1}{m}f'\right)(x_r) \end{pmatrix},$$

$A_0 \Longleftrightarrow$ Dirichlet boundary conditions $\qquad A_1 \Longleftrightarrow$ Neumann boundary conditions.

17.2 Outer system

In $\mathfrak{K}_l = L^2((-\infty, x_l))$ one defines

$$(T_l f)(x) := -\frac{1}{2m_l}\frac{d^2}{dx^2} f(x) + v_l(x) f(x),$$

$$\mathrm{dom}(T_l) := \left\{ f \in \mathfrak{K}_l : \begin{array}{l} f, \frac{1}{m_l} f' \in W^{1,2}((-\infty, x_l)) \\ f(x_l) = \left(\frac{1}{m_l} f'\right)(x_l) = 0 \end{array} \right\}.$$

Boundary triplet:

$$\Upsilon_0^l f := f(x_l) \quad \text{and} \quad \Upsilon_1^l f = -\left(\frac{1}{2m_l} f'\right)(x_l), \quad f \in \mathrm{dom}(T_l^*),$$

Weyl function:

$$\tau_l(\lambda) := i\sqrt{\frac{\lambda - v_l}{2m_l}}, \qquad \lambda \in \mathbb{C}_+.$$

In $\mathfrak{K}_r = L^2((x_r,\infty))$ one defines

$$(T_r f)(x) := -\frac{1}{2m_r}\frac{d^2}{dx^2}f(x) + v_r(x)f(x),$$

$$\mathrm{dom}(T_r) := \left\{ f \in \mathfrak{K}_r : \begin{array}{l} f, \frac{1}{m_r}f' \in W^{1,2}((x_r,\infty)) \\ f(x_r) = \left(\frac{1}{m_r}f'\right)(x_r) = 0 \end{array} \right\}.$$

Boundary triplet

$$\Upsilon_0^r f := f(x_r) \quad \text{and} \quad \Upsilon_1^r f = \left(\frac{1}{2m_r}f'\right)(x_r), \quad f \in \mathrm{dom}(T_r^*),$$

Weyl function:

$$\tau_r(\lambda) := i\sqrt{\frac{\lambda - v_r}{2m_r}}, \qquad \lambda \in \mathbb{C}_+.$$

Full outer system

$$\boldsymbol{L}^2(\mathbb{R} \setminus (\boldsymbol{x}_l, \boldsymbol{x}_r)) = \boldsymbol{L}^2((-\infty, \boldsymbol{x}_r)) \oplus \boldsymbol{L}^2((\boldsymbol{x}_r, \infty)),$$

$$\boldsymbol{T} = \boldsymbol{T}_l \oplus \boldsymbol{T}_r$$

Boundary triplet:

$$\boldsymbol{\Upsilon}_0 := \boldsymbol{\Upsilon}_0^l \oplus \boldsymbol{\Upsilon}_0^r \quad \text{and} \quad \boldsymbol{\Upsilon}_1 := \boldsymbol{\Upsilon}_1^l \oplus \boldsymbol{\Upsilon}_1^r$$

Weyl function:

$$\boldsymbol{\tau}(\boldsymbol{\lambda}) := \begin{pmatrix} \boldsymbol{\tau}_l(\boldsymbol{\lambda}) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\tau}_r(\boldsymbol{\lambda}) \end{pmatrix}$$

17.3 Strauss family

$$\mathrm{dom}(A_{-\tau(\lambda)}) := \left\{ f \in \mathfrak{H} : \begin{array}{c} f, \frac{1}{m}f' \in W^{1,2}((x_l, x_r)) \\ (\frac{1}{2m}f')(x_l) = -\tau_l(\lambda)f(x_l) \\ (\frac{1}{2m}f')(x_r) = \tau_r(\lambda)f(x_r) \end{array} \right\}, \quad \lambda \in \mathbb{C}_+,$$

and

$$(A_{-\tau(\lambda)}f)(x) = -\frac{1}{2}\frac{d}{dx}\frac{1}{m}\frac{d}{dx}f(x) + v(x)f(x), \quad x \in (x_l, x_r),$$

$f \in \mathrm{dom}(A_{-\tau(\lambda)})$, $\lambda \in \mathbb{C}_+$.

Characteristic function:

$$\Theta_{A_{-\tau(\lambda)}}(\mu) = I_{\mathcal{H}_{\tau(\lambda)}} - i\sqrt{2\,\Im\mathrm{m}(\tau(\lambda))}\Gamma_0(A^*_{-\tau(\lambda)} - \mu)^{-1}\Gamma_0^*\sqrt{2\,\Im\mathrm{m}(\tau_l(\lambda))}, \quad \mu \in \mathbb{C}_-.$$

for $\lambda \in \Sigma^\tau$.

17.4 Scattering

The coupled system coincides with the operator $\boldsymbol{L}$ defined at the beginning while the unperturbed operator coincides with $\boldsymbol{L}_0$, that is

$$\boldsymbol{L}_0 \boldsymbol{f} = -\frac{1}{2}\frac{d}{dx}\frac{1}{M}\frac{d}{dx}\boldsymbol{f} + V\boldsymbol{f},$$

with domain

$$\mathrm{dom}(\boldsymbol{L}_0) := \boldsymbol{W}_0^{2,2}((-\infty, x_l)) \oplus \left\{ \boldsymbol{f} \in \boldsymbol{W}^{1,2}((x_l, x_r)) : \begin{array}{c} \frac{1}{m}\boldsymbol{f}' \in \boldsymbol{W}^{1,2}((x_l, x_r)) \\ \boldsymbol{f}(x_l) = \boldsymbol{f}(x_r) = 0 \end{array} \right\} \oplus \boldsymbol{W}_0^{2,2}((x_r, \infty)).$$

Notice that $\boldsymbol{L}_0$ is the Friedrichs extension.

$\{\boldsymbol{L}.\boldsymbol{L}_0\}$ performs a complete scattering system, its scattering matrix is given by

$$S(\lambda) := \Theta_{A_{-\tau(\lambda)}}(\lambda - i0)^*.$$

17.5 R-matrix

We note that

$$\Re\mathrm{e}(\tau(\lambda)) = 0, \qquad \lambda \in (\max\{v_l, v_r\}, \infty).$$

Further

$$A_1 := A^* \upharpoonright \ker(\Gamma_1)$$

with

$$\ker(\Gamma_1) := \left\{ f \in W^{1,2}((x_l, x_r)) : \begin{array}{c} \frac{1}{m} f' \in W^{1,2}((x_l, x_r)) \\ \left(\frac{1}{2m} f'\right)(x_l) = \left(\frac{1}{2m} f'\right)(x_r) = 0 \end{array} \right\}.$$

The R-matrix admits the representation

$$R(\lambda) = \sum_{k \in \mathbb{N}} (\lambda_k - \lambda)^{-1} \left\langle \cdot, \begin{pmatrix} \sqrt{\Im\mathrm{m}(\tau_l(\lambda))}\psi_k(x_l) \\ \sqrt{\Im\mathrm{m}(\tau_r(\lambda))}\psi_k(x_r) \end{pmatrix} \right\rangle \begin{pmatrix} \sqrt{\Im\mathrm{m}(\tau_l(\lambda))}\psi_k(x_l) \\ \sqrt{\Im\mathrm{m}(\tau_r(\lambda))}\psi_k(x_r) \end{pmatrix}$$

for $\lambda \in (\max\{v_l, v_r\}, \infty)$ where λ_k and ψ_k are the eigenvalues and eigenfunctions of A_1.