On the relation between $XX^{[*]}$ and $X^{[*]}X$ in an indefinite inner product space.

Joint work with:

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Indefinite inner product

On $\mathbb{C}^n$ define indefinite inner product given by an invertible $H = H^*$ as follows:

$$[x, y] = \langle Hx, y \rangle$$

Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product.

Let X be an $n \times n$ matrix

H-adjoint:

$$[Xx, y] = [x, X^{[*]}y]$$

$$X^{[*]} = H^{-1}X^*H$$

H-selfadjoint matrices

An $n \times n$ matrix A is called *H-selfadjoint* if

$$HA = A^*H, \qquad [Ax, y] = [x, Ay].$$

Observation: both $X^{[*]}X$ and $XX^{[*]}$ are H-selfadjoint.

Examples

Define $P_k = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ of size $k \times k$ and $J_k(\lambda)$ the $k \times k$ Jordan block $J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & 1 \\ & & & \lambda \end{pmatrix}$

- $H = \varepsilon P_k, A = J_k(\lambda)$ with $\lambda \in \mathbb{R}, \varepsilon = \pm 1$
- $H = P_{2k}, A = J_k(\lambda) \oplus J_k(\bar{\lambda})$ with $\lambda \notin \mathbb{R}$

Canonical form

Theorem 1. *For any pair of matrices* (A, H) *with* $H = H^*$ *invertible and* $HA = A^*H$ *there exists an invertible matrix* S *such that* $(S^*HS, S^{-1}AS)$ *is a diagonal direct sum of blocks of the form as in the examples above.*

The signs ε*, one for each block corresponding to a real eigenvalue of* A*, are unique (up to trivialities).*

These signs are called the sign characteristic of the pair.

Problem statement

What can be said about the relations between the canonical forms for pairs $(XX^{[*]}, H)$ and $(X^{[*]}X, H)$?

Connection with *polar decomposition*. X allows H-polar decomposition if there are H-selfadjoint A and H-unitary U such that $X = UA$.

Surprise: X allows H-polar decomposition if and only if the canonical forms of $(XX^{[*]}, H)$ and $(X^{[*]}X, H)$ are the same.

Theorem of Flanders (1951)

The relations between the Jordan canonical forms of AB and BA are known.

Theorem 2. *The sizes of Jordan blocks of AB and BA corresponding to nonzero eigenvalues coincide.*
If AB has Jordan blocks with zero eigenvalue of sizes $n_1 \geq n_2 \geq ...$ made infinitely with adjunction of zeros, and $m_1 \geq m_2 \geq ...$ is the corresponding sequence of sizes of Jordan blocks of BA with zero as eigenvalue, then $\|n_j - m_j\| \leq 1$.

Consequently: sizes of Jordan blocks with zero eigenvalue can go up by one, down by one, or stay the same, but the algebraic multiplicity of zero as eigenvalue is the same for AB and BA.

Nonzero eigenvalues of $X^{[*]}X$ and $XX^{[*]}$

Clearly, for nonreal eigenvalues the blocks in the canonical forms of $(X^{[*]}X, H)$ and $(XX^{[*]}, H)$ coincide. For real eigenvalues we need to consider the sign characteristic. For nonzero eigenvalues this is easy:

Proposition 1. *If $\lambda_1 \neq 0$ is a real eigenvalue of $X^{[*]}X \in \mathbb{C}^{n\times n}$ with the corresponding sign ε_1 attached to the Jordan block $J(\lambda_1)$ in the canonical form for $(X^{[*]}X, H)$, then the corresponding sign attached to this Jordan block in the canonical form for $(XX^{[*]}, H)$ is* $\operatorname{sign}(\lambda_1)\varepsilon_1$.

Idea of proof

Let $\mathbf{x}_k, \mathbf{x}_{k-1}, \ldots, \mathbf{x}_0$, be a Jordan chain of $X^{[*]}X$, corresponding to the nonzero real eigenvalue λ.
Put $\mathbf{y}_j = X\mathbf{x}_j$ for $j = 0, \ldots, k$.
Then $\mathbf{y}_k, \mathbf{y}_{k-1}, \ldots, \mathbf{y}_0$ is a Jordan chain of $XX^{[*]}$.

The sign attached to the Jordan block corresponding to λ in the canonical form for $(X^{[*]}X, H)$ is equal to the sign of $[\mathbf{x}_0, \mathbf{x}_k] = \langle H\mathbf{x}_0, \mathbf{x}_k \rangle$.
The sign attached to the corresponding block in the canonical form for $(XX^{[*]}, H)$ is given by

$$\langle H\mathbf{y}_0, \mathbf{y}_k \rangle = \langle HX\mathbf{x}_0, X\mathbf{x}_k \rangle = \langle X^*HX\mathbf{x}_0, \mathbf{x}_k \rangle =$$

$$\left\langle HX^{[*]}X\mathbf{x}_0, \mathbf{x}_k \right\rangle = \lambda \langle H\mathbf{x}_0, \mathbf{x}_k \rangle .$$

The zero eigenvalue

Conclusion: we may assume that $X^{[*]}X$ and $XX^{[*]}$ are nilpotent. At present the relations between the canonical forms of $(X^{[*]}X, H)$ and $(XX^{[*]}, H)$ are only known in some special cases.

- rank $X^{[*]}X =$ rank X,
- rank $X = 1$ or rank $X = n - 1$,
- $X^{[*]}X = 0$.

Observe: the first and last case are in a way opposite extremes: Ker $X \subset$ Ker $X^{[*]}X$, so rank $X^{[*]}X \leq$ rank X.

The case rank $X^{[*]}X =$ rank X

Proposition 2. *Let X be a nilpotent matrix such that rank $X^{[*]}X =$ rank X. Then $(X^{[*]}X, H)$ and $(XX^{[*]}, H)$ cannot have the same canonical form unless $X = 0$.*

Proof. Suppose $(X^{[*]}X, H)$ and $(XX^{[*]}, H)$ have the same canonical form. Then X admits an H-polar decomposition: $X = UA$, with an H-unitary U and an H-selfadjoint A. Clearly Ker $X =$ Ker A and $A^2 = X^{[*]}X$.
We have Ker $X =$ Ker $X^{[*]}X$, and so:

$$\text{Ker } A = \text{Ker } X = \text{Ker } X^{[*]}X = \text{Ker } A^2.$$

The matrix A is also nilpotent, so $A = 0$, and hence $X = 0$. $\square$

Corollary 1. *If* rank $X^{[*]}X =$ rank X *and $X \neq 0$ then X does not allow an H-polar decomposition.*

The case rank $X^{[*]}X =$ rank X, continued

Theorem 3. *Assume that X is a matrix for which* Ker $X^{[*]}X =$ Ker X. *Let the canonical form of $(X^{[*]}X, H)$ be given by*

$$\oplus_{j=1}^{k} J_{n_j} \oplus \ \oplus_{j=k+1}^{l} J_1, \quad \oplus_{j=1}^{k} \varepsilon_j P_{n_j} \oplus \ \oplus_{j=k+1}^{l} \varepsilon_j,$$

where we assume that $n_j > 1$ for $j = 1, \dots, k$. Then the canonical form of $(XX^{[*]}, H)$ is given by

$$\oplus_{j=1}^{k} J_{n_j-1} \oplus \oplus_{j=1}^{k} J_1 \oplus \oplus_{j=k+1}^{l} J_1, \quad \oplus_{j=1}^{k} \varepsilon_j P_{n_j-1} \oplus \oplus_{j=1}^{k} \delta_j \oplus \oplus_{j=k+1}^{l} \varepsilon_j,$$

and the numbers $\delta_j = \pm 1$ are determined by the equation one obtains from comparing the signature of H in both canonical forms:

$$\sum_{n_j \text{ is odd}} \varepsilon_j = \sum_{n_j \text{ is even}} \varepsilon_j + \sum_{j=1}^{k} \delta_j.$$

Corollaries I: rank $X = 1$

There are only three possibilities:

i $X^{[*]}X = XX^{[*]} = 0$. This is a trivial case.

ii rank $X^{[*]}X = 1$ and $XX^{[*]} = 0$. Apply the Theorem directly.

iii rank $XX^{[*]} = 1$ and $X^{[*]}X = 0$. Apply the Theorem for X and $X^{[*]}$ interchanged.

Corollaries II: rank $X^{[*]}X = n-1$

Ker $X^{[*]}X$ is one-dimensional. Let ε be the (single) sign in the sign characteristic of $(X^{[*]}X, H)$. Then the Jordan canonical form of $XX^{[*]}$ consists of one block of size $n-1$ and one block of size one.

The signs in the sign characteristic of $(XX^{[*]}, H)$ are as follows.
In case n is odd both signs are ε.
In case n is even,
the sign corresponding to the block of size $n-1$ is ε,
and the sign corresponding to the block of size 1 is $-\varepsilon$.

The case rank $X = n-1$

First observation: if either rank $X^{[*]}X = n-1$, or rank $XX^{[*]} = n-1$ we can apply the previous theorem.
So we may assume that

$$\text{dim Ker } X^{[*]}X = \text{dim Ker } XX^{[*]} = 2.$$

Assume that

$$X^{[*]}X = J_k \oplus J_{n-k}, \quad H = \varepsilon_1 P_k \oplus \varepsilon_2 P_{n-k}, \quad k \le n-k.$$

Denote X by $X = \begin{pmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{pmatrix}$.
Since Ker X is one-dimensional there are complex numbers α and β, not both zero, such that Ker $X = \text{span}\,\{\alpha \mathbf{e}_1 + \beta \mathbf{e}_{k+1}\}$, i.e.,

$$\alpha \mathbf{x}_1 + \beta \mathbf{x}_{k+1} = 0.$$

The case rank $X = n - 1$ continued

Theorem 4. *With this notation the following hold:*

a *if* $\mathbf{x}_{k+1} = 0$ (*and so* $\alpha = 0$), *then* $XX^{[*]} \approx J_{k+1} \oplus J_{n-k-1}$ *with corresponding signs* ε_1 *and* ε_2.

b *if* $\mathbf{x}_{k+1} \neq 0$ *then there are the following possibilities*

i. $2k \neq n$. *In this case* $XX^{[*]} \approx J_{k-1} \oplus J_{n-k+1}$ *with corresponding signs* ε_1 *and* ε_2.

ii. $2k = n, \varepsilon_1 = \varepsilon_2$. *In this case* $XX^{[*]} \approx J_{k-1} \oplus J_{k+1}$ *with corresponding signs both equal to* ε_1.

iii. $2k = n, \varepsilon_1 = -\varepsilon_2$ *and* $|\alpha| \neq |\beta|$. *Then* $XX^{[*]} \approx J_{k-1} \oplus J_{k+1}$ *with corresponding signs*
sign $(|\alpha|^2 - |\beta|^2)\varepsilon_1$ *and* sign $(|\alpha|^2 - |\beta|^2)\varepsilon_2$.

iv. $2k = n, \varepsilon_1 = -\varepsilon_2$ *and* $|\alpha| = |\beta|$. *Then* $XX^{[*]} \approx J_k \oplus J_k$ *with corresponding signs* $+1$ *and* -1.

The case $X^{[*]}X = 0$

In this case Im X is H-neutral.
Let $\mathcal{N}_0$ be a subspace that is skewly linked to Im X, and let $\mathcal{N}_1$ be the subspace $(\text{Im } X \dot{+} \mathcal{N}_0)^{[\perp]}$. Then $\mathcal{N}_1$ is H-nondegenerate.
Decompose $\mathbb{C}^n = \text{Im} X \dot{+} \mathcal{N}_0 \dot{+} \mathcal{N}_1$ then

$$X = \begin{pmatrix} X_1 & X_2 & X_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & H_3 \end{pmatrix}.$$

Note X_1 is nilpotent.
Denote $j = \dim \text{Im } X$ then

$$\text{rank } \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix} = j.$$

The case $X^{[*]}X = 0$ continued

Then

$$XX^{[*]} = \begin{pmatrix} 0 & Z & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$Z = X_1 X_2^* + X_2 X_1^* + X_3 H_3^{-1} X_3^*.$$

The Jordan canonical form of $XX^{[*]}$ depends on the rank of Z. For instance, if Z happens to be full rank then there are j Jordan chains of lenght two, while if $Z = 0$ then $XX^{[*]} = 0$.
Denote rank $Z = k$.

The case $X^{[*]}X = 0$, final result

Theorem 5. *Assume that $X^{[*]}X = 0$, let X and H and Z be as above. Let $k = \text{rank } Z$, and $j = \dim \text{ Im } X$. Let*
$\kappa_+ =$ the number of positive eigenvalues of H_3,
$\kappa_- =$ the number of negative eigenvalues of H_3,
$\nu_+ =$ the number of positive eigenvalues of Z,
$\nu_- =$ the number of negative eigenvalues of Z.

Then the Jordan canonical form of $XX^{[*]}$ has $n - 2k$ Jordan blocks of size one, and k Jordan blocks of size two.
The signs in the sign characteristic of $(XX^{[*]}, H)$ corresponding to the Jordan blocks of size one are as follows: the number of $+1$'s is $\kappa_+ + (j - k)$ and the number of -1's is $\kappa_- + (j - k)$.
The signs in the sign characteristic of $(XX^{[*]}, H)$ corresponding to the Jordan blocks of size two are as follows: the number of $+1$'s is ν_+, while the number of -1's is ν_-.