Composition of Dirac structures for infinite–dimensional systems

Adrian Sandovici

joint work with Orest Iftime

Outline

Dirac structures on real vector spaces

Let $\mathcal{F}$ and $\mathcal{E}$ be real vector spaces whose elements are labeled as f and e, respectively.

The space $\mathcal{F}$ is called the space of *flows*

The space $\mathcal{E}$ is called the space of *efforts*.

The space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ is called the *bond space* and an element of the space $\mathcal{B}$ is denoted by $b = (f, e)$.

The spaces $\mathcal{F}$ and $\mathcal{E}$ are power conjugate. This means that there exists a map

$$\langle \cdot \mid \cdot \rangle : \mathcal{E} \times \mathcal{F} \longrightarrow \mathbb{R}$$

called the *power product* such that it is linear in each coordinate and it is not degenerate.

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Using the power product define a symmetric *bilinear form*

$$\ll \cdot, \cdot \gg : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$$

by

$$\ll (f^1, e^1), (f^2, e^2) \gg = \langle e^1 \mid f^2 \rangle + \langle e^2 \mid f^1 \rangle,$$

for all $(f^1, e^1), (f^2, e^2) \in \mathcal{B}$.

We have the following immediate relation between the power product and and the bilinear form

$$\langle e \mid f \rangle = \frac{1}{2} \ll b, b \gg$$

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Definition: Tellegen structure

Let $\mathcal{Z}$ be a subspace of the vector space $\mathcal{B}$. $\mathcal{Z}$ is a *Tellegen structure* on $\mathcal{B}$ if

$$\langle e \mid f \rangle = 0, \;\; \forall\, (f, e) \in \mathcal{Z}.$$

Denote $\mathcal{Z}^{\perp}$ the orthogonal complement of $\mathcal{Z}$ with respect to the bilinear form $\ll \cdot, \cdot \gg$

$$\mathcal{Z}^{\perp} := \{ b \in \mathcal{B} \mid \ll b, \tilde{b} \gg = 0, \; \forall\, \tilde{b} \in \mathcal{Z} \}.$$

Let $\mathcal{Z}$ be a subspace of the vector space $\mathcal{B}$. Then $\mathcal{Z}$ is a Tellegan structure on $\mathcal{B}$ if and only if

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For finite-dimensional spaces a Dirac structure is a Tellegan structure of maximal dimension.

In what follows the focus will be on the case when $\mathcal{B}$ is a reflexive Banach space.

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In what follows the focus will be on the case when $\mathcal{B}$ is a reflexive Banach space.

Let $\mathcal{F}$ be a (real) Banach space and $\mathcal{E} = \mathcal{F}^*$, where $\mathcal{F}^*$ is the adjoint space of $\mathcal{F}$ (the set of all bounded semi-linear forms on $\mathcal{F}$). Then $\mathcal{E}$ is a Banach space with the norm $\|e\|$ defined by

$$\|e\| = \sup_{0 \neq f \in \mathcal{F}} \frac{|e(f)|}{\|f\|}.$$

Assumption

The Banach space $\mathcal{F}$ is reflexive.

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Introduce the scalar product

$$\langle \cdot \mid \cdot \rangle_{\mathcal{E}\times\mathcal{F}} : \mathcal{E} \times \mathcal{F} \to \mathbb{R}$$

defined by

$$\langle e \mid f \rangle_{\mathcal{E}\times\mathcal{F}} := e(f)$$

for all $e \in \mathcal{E}$ and $f \in \mathcal{F}$.

Each $f \in \mathcal{F}$ may be regarded as an element of $\mathcal{F}^{**}$ and introduce in a similar way another scalar product

$$\langle \cdot \mid \cdot \rangle_{\mathcal{F}\times\mathcal{E}} : \mathcal{F} \times \mathcal{E} \to \mathbb{R}$$

which will satisfy $\langle f \mid e \rangle_{\mathcal{F}\times\mathcal{E}} = \langle e \mid f \rangle_{\mathcal{E}\times\mathcal{F}}$. for all $e \in \mathcal{E}$ and $f \in \mathcal{F}$.

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The scalar product on $\mathcal{E} \times \mathcal{F}$ is a power product.

The bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ is also a reflexive Banach space with the linear structure defined componentwise and the norm defined by

$$\|(f, e)\| = (\|f\|^2 + \|e\|^2)^{\frac{1}{2}}.$$

Consider the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{B}^* \times \mathcal{B}} : \mathcal{B}^* \times \mathcal{B} \to \mathbb{R}$ given by

$$\langle b, \tilde{b} \rangle_{\mathcal{B}^* \times \mathcal{B}} := \langle e \mid \tilde{f} \rangle + \langle \tilde{e} \mid f \rangle$$

where $b = (e, f) \in \mathcal{B}^* = \mathcal{E} \times \mathcal{F}$ and $\tilde{b} = (\tilde{f}, \tilde{e}) \in \mathcal{B} = \mathcal{F} \times \mathcal{E}$.

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For any subset $\mathcal{Z}$ of $\mathcal{B}$ denote by $\mathcal{Z}^c$ the orthogonal complement with respect to the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{B}^* \times \mathcal{B}}$, i.e.

$$\mathcal{Z}^c := \{ b \in \mathcal{B}^* \mid \langle b, \tilde{b} \rangle_{\mathcal{B}^* \times \mathcal{B}} = 0, \ \forall \tilde{b} \in \mathcal{Z} \}.$$

$\mathcal{Z}^c$ is a subset of $\mathcal{B}^* = \mathcal{E} \times \mathcal{F}$ and $\mathcal{Z}^\perp$ (the orthogonal of $\mathcal{Z}$ with respect to the bilinear form on $\mathcal{B}$ as defined in the previous section)

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Consider R the natural embedding of $\mathcal{B}^*$ into $\mathcal{B}$. Then R is an isometric isomorphism between $\mathcal{B}^*$ and $\mathcal{B}$ defined by

$$R = \begin{bmatrix} 0 & r_{\mathcal{F}^{**}\mathcal{F}} \\ id_{\mathcal{E}} & 0 \end{bmatrix}$$

where $id_{\mathcal{E}}$ is the identity on $\mathcal{E}$ and $r_{\mathcal{F}^{**}\mathcal{F}}$ is the inverse of $r_{\mathcal{F}\mathcal{F}^{**}}$, the natural isometric isomorphism between $\mathcal{F}$ and $\mathcal{F}^{**}$.

The inverse of R is the isometric isomorphism $S : \mathcal{B} \longrightarrow \mathcal{B}^*$ defined by

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The bilinear form $\ll \cdot, \cdot \gg$ on $\mathcal{B}$ is related to the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{B}^* \times \mathcal{B}}$ by

$$\ll b^1, b^2 \gg = \langle Sb^1, b^2 \rangle_{\mathcal{B}^* \times \mathcal{B}}.$$

for all $b^1 = (f^1, e^1)$ and $b^2 = (f^2, e^2)$ in $\mathcal{B}$.

Proposition

Let $\mathcal{Z}$ be a subspace of the bond space $\mathcal{B}$. Then the following equalities holds:

$$\mathcal{Z}^\perp = R\mathcal{Z}^c, \quad S\mathcal{Z}^\perp = \mathcal{Z}^c.$$

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Let D be a vectorial subspace of $\mathcal{B}$. The following statements are equivalent:

1. D is a Dirac structure on $\mathcal{B}$.
2. $D = RD^c$.
3. $D^c = SD$.

Theorem

Let $\mathcal{Z}$ be a closed subspace of the bond space $\mathcal{B}$. Then $\mathcal{Z}$ is a Dirac structure on $\mathcal{B}$ if and only if $\mathcal{Z}$ and $\mathcal{Z}^\perp$ are Tellegan structures on $\mathcal{B}$.

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An example

Let A be a skew-adjoint (unbounded in general) operator from $domA \subseteq \mathcal{F}$ to $\mathcal{E}$, that is

$$\langle Ax \mid y \rangle + \langle x \mid Ay \rangle = 0,$$

for all $x, y \in domA = domA^*$.

Then the graph of A,

$$\mathcal{G}(A) = \{(x, Ax) \ : \ x \in domA\}$$

is a Dirac structure.

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Consider a densely defined closed operator $T : \mathcal{B} \rightarrow \mathcal{L}$, where $\mathcal{B}$ is the bond space. The subspace $D = \ker\,(T)$ is a Dirac structure on $\mathcal{B}$ if and only if $\ker\,(T)$ and $\operatorname{Im}(RT^{*})$ are Tellegan structures on $\mathcal{B}$.

Theorem

Let D be a Dirac structure on the bond space $\mathcal{B}$. There exists a projection P from $\mathcal{B}$ onto D if and only if $\mathcal{B} = D \oplus N$ for some closed subspace N of $\mathcal{B}$. Then

$$D = \ker\,(I - P).$$

The kernel representation of a Dirac structure

Consider a densely defined closed operator $T : \mathcal{B} \to \mathcal{L}$, where $\mathcal{B}$ is the bond space. The subspace $D = \ker\,(T)$ is a Dirac structure on $\mathcal{B}$ if and only if $\ker\,(T)$ and $\operatorname{Im}(RT^*)$ are Tellegan structures on $\mathcal{B}$.

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Consider a transmission line whose length is S. The Kirchhoff's laws describing the transmission line are given by

$$e_\phi = -\frac{\partial e_q}{\partial z},$$
$$f_q = -\frac{\partial f_\phi}{\partial z}.$$

Here f_q is the rate of charge density, e_q is the voltage distribution, f_ϕ is the current distribution and e_ϕ is the rate of flux density.

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The kernel representation of a Dirac structure

The boundary conditions are

$$f_\phi(0) = -f_L,\ e_q(0) = e_L,$$
$$f_\phi(S) = f_R,\ e_q(S) = e_R.$$

Here f_L and e_L are the current and voltage at the left boundary.

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Let p, q be two positive numbers satisfying the condition $1/p + 1/q = 1$ and let $L_p(0, S)$ and $L_q(0, S)$ be the space of p- and q-integrable functions on $[0, S]$, respectively.

The space of flow variables is given by

$$\mathcal{F} = L_p(0, S) \times L_p(0, S) \times \mathbb{R}^2,$$

while the space of effort variables is given by

$$\mathcal{E} = L_q(0, S) \times L_q(0, S) \times \mathbb{R}^2.$$

An element of the space $\mathcal{F}$ is denoted by $f = (f_q, f_\phi, f_L, f_R)$, and an element of the space $\mathcal{E}$ is denoted by $e = (e_q, e_\phi, e_L, e_R)$.

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The power product is defined as

$$\begin{aligned}\langle e|f\rangle_{\mathcal{B}} &= \langle e,f\rangle_{\mathcal{F}} \\ &= \int_0^S f_q(z)e_q(z)dz + \int_0^S f_\phi(z)e_\phi(z)dz \\ &\quad + f_L e_L + f_R e_R.\end{aligned}$$

The first term represents the power associated to electrical domain, the second term is power associated to magnetic domain and the last two terms represents the power exchanged through the boundary.

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The kernel representation of a Dirac structure

The space of admissible flows and efforts is given by

$$\mathcal{D} = \ker\,(T),$$

where
$T : \mathcal{B} \to \mathcal{L} = L_p(0,S) \times L_p(0,S) \times \mathbb{R}^2 \times L_q(0,S) \times L_q(0,S) \times \mathbb{R}^2$ is as follows

$$T = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix},$$

with M and N given by:

The kernel representation of a Dirac structure

The space of admissible flows and efforts is given by

$$\mathcal{D} = \ker\,(T),$$

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$$M = \begin{bmatrix} id_{L_p(0,S)} & \frac{\partial}{\partial z} & 0 & 0 \\ 0 & \partial_{L,p} & 1 & 0 \\ 0 & \partial_{R,p} & 0 & -1 \end{bmatrix},$$

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The domain of the operator T is

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The decomposition of a Dirac structure

Three classes of Dirac structures are introduced:

1. *Completely multi–valued Dirac structures* which are of the form
$$\mathcal{D}_{mul} = \{(0, e) \,:\, e \in \mathcal{E}\};$$
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Let $\mathcal{F}$ be a Hilbert space, $\mathcal{E} = \mathcal{F}^*$ the dual of $\mathcal{F}$, and consider $R : \mathcal{F} \to \mathcal{E}$ an isometric isomorphism between $\mathcal{F}$ and $\mathcal{E}$.

For any linear subspace $\mathcal{V}$ of $\mathcal{B}$ define the linear relation $\mathcal{O}$ in $\mathcal{E}$ by

$$\mathcal{O}_{\mathcal{V}} = I_{\mathcal{E}} - 2R(\mathcal{V} + R)^{-1}.$$

Let $\mathcal{D}$ be a Dirac structure on the Hilbert space $\mathcal{B}$. Then $\mathcal{O}_{\mathcal{D}}$ is a unitary operator in $\mathcal{E}$.

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The composition of Dirac structures

Let $\mathcal{F}_i$, $1 \leq i \leq 3$ be three Hilbert spaces, let $\mathcal{E}_i = \mathcal{F}_i^*$, $1 \leq i \leq 3$, and let $R_i : \mathcal{F} \to \mathcal{E}$, $1 \leq i \leq 3$ be the corresponding isometric isomorphisms.

Consider two Dirac structures $\mathcal{D}_A$ and $\mathcal{D}_B$ on $\mathcal{F}_1 \times \mathcal{E}_1 \times \mathcal{F}_2 \times \mathcal{E}_2$ and on $\mathcal{F}_2 \times \mathcal{E}_2 \times \mathcal{F}_3 \times \mathcal{E}_3$, respectively.

$\mathcal{D}_A$ and $\mathcal{D}_B$ have the following scaterring representations:

$$\mathcal{O}_A = \begin{bmatrix} \mathcal{O}_{11}^A & \mathcal{O}_{12}^A \\ \mathcal{O}_{21}^A & \mathcal{O}_{22}^A \end{bmatrix}, \quad \mathcal{O}_B = \begin{bmatrix} \mathcal{O}_{22}^B & \mathcal{O}_{23}^B \\ \mathcal{O}_{32}^B & \mathcal{O}_{33}^B \end{bmatrix}$$

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The composition $\mathcal{D}_A \circ \mathcal{D}_B$ of $\mathcal{D}_A$ and $\mathcal{D}_B$ consists of all (f_1, e_1, f_3, e_3) such that $(f_1, e_1, f_2, e_2) \in \mathcal{D}_A$ and $(f_2, e_2, f_3, e_3) \in \mathcal{D}_B$ for some $(f_2, e_2) \in \mathcal{F}_2 \times \mathcal{E}_2$.

The composition of two Dirac structures is always a Tellegen structure.

For finite–dimensional systems, the composition of two Dirac structures defines a Dirac structure. For distributed–parameter systems the composition of two Dirac structures does not always result in a new Dirac structure.

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When is the composition again a Dirac structure?

Theorem

Let $\mathcal{D}_A$ and $\mathcal{D}_B$ be two Dirac structures and let $\mathcal{O}_A$ and $\mathcal{O}_B$ the corresponding scattering representations of them. The following items are equivalent:

1. $\mathcal{D}_A \circ \mathcal{D}_B$ is a Dirac structure;
2. $\mathcal{O}_B \star \mathcal{O}_A$ is a unitary operator in $\mathcal{E}_1 \times \mathcal{E}_3$;
3. The following conditions are both met:
   1. $\operatorname{ran} \begin{bmatrix} \mathcal{O}^A_{21} & \mathcal{O}^A_{22}\mathcal{O}^B_{23} \end{bmatrix} \subset \operatorname{ran} \left(\mathcal{O}^A_{22}\mathcal{O}^B_{22} - I_{\mathcal{E}_2} \right)$, and
   2. $\operatorname{ran} \begin{bmatrix} \mathcal{O}^{B*}_{22}\mathcal{O}^{A*}_{12} & \mathcal{O}^{B*}_{32} \end{bmatrix} \subset \operatorname{ran} \left(\mathcal{O}^{B*}_{22}\mathcal{O}^{A*}_{22} - I_{\mathcal{E}_2} \right)$.

In particular, if $I_{\mathcal{E}_2} - \mathcal{O}^A_{22}\mathcal{O}^B_{22}$ is invertible in $\mathcal{E}_2$ then $\mathcal{D}_A \circ \mathcal{D}_B$ is a Dirac structure on $\mathcal{F}_1 \times \mathcal{E}_1 \times \mathcal{F}_3 \times \mathcal{E}_3$.

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