Remarks on Global Stability for Inverse Sturm–Liouville problems

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Borg, 1946

$$L_D = \begin{cases} Ly = -y'' + q(x)y \\ \mathcal{D}(L) = \{y \in W_2^2 \mid y(0) = y(\pi) = 0\} \end{cases}$$

$$L_{DN} = \begin{cases} Ly = -y'' + q(x)y \\ \mathcal{D}(L) = \{y \in W_2^2 \mid y(0) = y'(\pi) = 0\} \end{cases}$$

Denote by $\{\lambda_k\}_1^\infty$ and $\{\mu_k\}_1^\infty$ the spectra of L_D and L_{DN}, respectively.

Direct problem:

$$\rho_{2k-1} = \sqrt{\mu_k} = k - \frac{1}{2} + \frac{q_0}{2k-1} + \frac{\alpha_{2k-1}}{k}$$
$$\rho_{2k} = \sqrt{\lambda_k} = k - \frac{q_0}{2k} + \frac{\alpha_{2k}}{k}$$

where q_0 is the mean value of q and $\{\alpha_k\}_1^\infty \in \ell_2$. Also,

$$\rho_1 < \rho_2 < \ldots < \rho_n < \rho_{n+1} \ldots \qquad (IC)$$

Inverse Problem (Borg, Marchenko, Levitan). Let

$$s_{2k-1} = \rho_{2k-1} - k + \frac{1}{2}, \qquad s_{2k} = \rho_{2k} - k.$$

Assume

$$\{s_k\}_1^\infty \in \ell_2^1 \qquad \left(\sum |s_k|^2 k^2 < \infty\right).$$

and the condition (IC) holds. Then $\exists!\ q(x) \in L_2(0,\pi)$ such that $\{\lambda_k\}$ and $\{\mu_k\}$ are the spectra of L_D and L_{DN}

There is a recovery procedure. It turns over that from two spectra we can recover the norming constants (by Levinson formulas)

$$\alpha_n = \int_0^\pi s^2(x,\lambda_n)\,dx,$$

where s is the solution of $-y'' + q(x)y = \lambda y$ satisfying the conditions

$$s(0,\lambda) = 0, \quad s'(0,\lambda) = 1$$

Then the potential q can be found by Gelfand-Levitan method.

Local stability (Marchenko, McLaughlin, Yurko, Malamud).

Theorem. *Let the sequences* $\{\rho_k\}_1^\infty$, $\{\tilde\rho_k\}_1^\infty$, *correspond to the potentials* $q(x)$ *and* $\tilde q(x)$. *Then*

$$\|q-\tilde q\|_{L_2}^2 \leqslant C\sum_{k=1}^{\infty}|\rho_k-\tilde\rho_k|^2\,k^2 = C\sum_{k=1}^{\infty}|s_k-\tilde s_k|^2\,k^2$$

where $C=C(q,\varepsilon)$, *provided that* $\|q-\tilde q\|\leqslant\varepsilon$.

We want to get

$$\|q-\tilde q\|_{L_2}^2 \leqslant C\sum_{k=1}^{\infty} |\rho_k-\tilde\rho_k|^2\, k^2 = C\sum_{k=1}^{\infty} |s_k-\tilde s_k|^2\, k^2 \qquad (1)$$

for all $\{\beta_k\}$, $\{\tilde\beta_k\}$ such that

$$\{s_k\},\ \{\tilde s_k\} \in B_r \quad = \text{the ball of radius } r \text{ in } \ell_2^1, \qquad (2)$$

with $C = C(r)$.

This is not true. However, this is true with $C = C(r,h)$, provided that $\{s_k\}$ and $\{\tilde s_k\}$ obey the additional assumption

$$|\rho_k-\rho_{k+1}| \geqslant h, \qquad |\tilde\rho_k-\tilde\rho_{k+1}| \geqslant h. \qquad (IC)_h$$

Theorem. *Let (2) and $(IC)_h$ hold. Then (1) holds with C depending only on r and h.*

To prove this result we have to solve more general problem: to give a complete characterization of spectral data $\{\rho_k\}_1^\infty$ for potentials $q(x) \in W_2^\theta$. We do this for $-1 \leqslant \theta < \infty$.

Remind

$$y^{[1]} = y'(x) - \sigma(x)y,$$

$$\sigma(x) = \int q(x)\,dx \qquad \text{(in the sense of distibution)},$$

$$L(y) = (y^{[1]})' - \sigma(x)y^{[1]} - \sigma^2(x)y,$$

then L_D and L_{DN} are well defined for $q(x) \in W_2^\theta$, $\theta \geqslant -1$, $\sigma \in W_2^{\theta+1}$.

Define

$$\ell_2^\theta = \{\{s_k\}_1^\infty | \quad \sum k^{2\theta}|s_k|^2 < \infty\}.$$

Constract the spaces

$$\widehat{\ell}_2^\theta = \ell_2^\theta, \qquad 0 \leqslant \theta < 1/2$$

$$\widehat{\ell}_2^\theta = \ell_2^\theta \oplus \left\{\frac{1}{k}\right\} \oplus \left\{\frac{(-1)^k}{k}\right\}, \qquad \frac{1}{2} \leqslant \theta < \frac{5}{2},$$

$$\widehat{\ell}_2^\theta = \ell_2^\theta + \left\{\frac{1}{k}\right\} \oplus \left\{\frac{(-1)^k}{k}\right\} + \left\{\frac{1}{k^3}\right\} + \left\{\frac{(-1)^k}{k^3}\right\}, \quad \frac{5}{2} \leqslant \theta < \frac{9}{2},$$

. . . .

Lemma. *Let*

$$T\sigma = \{b_k\}_1^\infty, \qquad b_k = \frac{2}{\pi}\int_0^\pi \sigma(x)\sin kx\,dx.$$

Then $T : W_2^\theta \to \widehat{\ell}_2^\theta$ *is an isomorphism for all* $\theta \geqslant 0$*.*

Theorem. *Let*

$$F(\sigma) = \{s_k\}_1^\infty.$$

Then $F : W_2^\theta \to \widehat{\ell}_2^\theta$ *and for all* $\theta > 0$

$$F\sigma = \frac{1}{2}T\sigma + \Phi(\sigma),$$

where $\Phi : W_2^\theta \to \widehat{\ell}_2^\tau$*,*

$$\tau = \begin{cases} 2\theta, & \textit{if } 0 \leqslant \theta \leqslant 1 \\ \theta + 1, & \textit{if } 1 \leqslant \theta < \infty, \end{cases}$$

and

$$\|\Phi(\sigma)\|_\tau \leqslant C\|\sigma\|_\theta$$

for all $\sigma \in B_R(W_2^\theta)$ *with* $C = C(R)$*.*

Hence, **F is a weakly nonlinear map!!!**

We do not know if this result is true for $\theta = 0$.

Proved in

A.M.Savchuk and A.A.Sh // Math. Notes **80**, No 6 (2006)

Theorem. *$F : \sigma \to \{s_k\}_1^\infty$ is a bijection from W_2^θ into Σ^θ,*

$$\Sigma^\theta = \left\{ \{s_k\} \in \widehat{\ell}_2^\theta \mid \quad (IC)\ holds \right\}.$$

This gives a complete solution of the inverse problem in the whole scale of Sobolev spaces W_2^θ, $\theta \geqslant 0$ (in terms if $\sigma(x)$).

It is sufficient to prove that:

a) F is injective for $\theta = 0$,

b) F is surjective for $0 \leqslant \theta < \varepsilon$.

Proved in

A.M.Savchuk and A.A.Sh // Russian J. Math. Ph. **12** No 4 (2005) (in Levitan memorial volume).

For $0 \leqslant \theta \leqslant 1$ in the other terms independently by R.O.Hryniv and Ya.V.Mykytyuk // Proc.Royal Soc. Eddinburg (2006), arXiv math FA/0406238 (2005).

Let

$$\Sigma^{\theta}_{r,h} = \left\{ \{s_k\}_1^{\infty} \in \widehat{\ell}^{\theta}_2 \mid \quad \|\{s_k\}\|_{\widehat{\ell}^{\theta}} \leqslant r, \quad (IC)_h \text{ holds} \right\},$$

$$|\rho_{k+1} - \rho_k| > h \qquad \forall \ k \geqslant 1 \qquad\qquad (IC)_h$$

Lemma. $F^{-1}(\Sigma^{\theta}_{r,h}) \subset B_R(W^{\theta}_2)$, *where* $R = R(r,h)$.

Theorem (on global stability). *For all* $\theta > 0$ *(*$= 0$*?) the estimate*

$$\|\sigma - \tilde{\sigma}\|^2_{W^\theta} \leqslant C\|\{\rho_k\} - \{\tilde{\rho}_k\}\|_{\widehat{\ell}^\theta} = C\|\{s_k\} - \{\tilde{s}_k\}\|_{\widehat{\ell}^\theta}$$

holds provided that

$$\{s_k\},\ \{\tilde{s}_k\} \in \Sigma^\theta_{r,h},$$

and $C = C(\theta, r, h)$.

The inverse estimate, certainly, also holds.