A test for commutative J-symmetric families of D_κ^+-class.

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Abstract

A goal of this report is a study of relations between commutative J-symmetric families of so-called D_κ^+-class and some spectral functions with peculiarities.

1. Preliminaries

Let $\mathcal{H}$ be a Hilbert space. If $\mathcal{Y} \subseteq \mathcal{H}$, then the symbol $\operatorname{Lin}\mathcal{Y}$ refers to the linear span of $\mathcal{Y}$, and by the symbol $\operatorname{CLin}\mathcal{Y}$ we denote the closed linear span of $\mathcal{Y}$. The symbol $\dim \mathcal{X}$ is the linear dimension of a vector space $\mathcal{X}$. In what follows $\mathcal{H}$ is a Krein space with an indefinite sesquilinear form $[\cdot,\cdot]$. Let $\mathcal{H} = \mathcal{H}_+[\dot{+}]\mathcal{H}_-$ be a canonical decomposition of $\mathcal{H}$, let P_+ and P_- be canonical projections: $\mathcal{H}_+ = P_+\mathcal{H}$, $\mathcal{H}_- = P_-\mathcal{H}$, let $J = P_+ - P_-$ be a canonical symmetry, and let $(\cdot,\cdot) = [J\cdot,\cdot]$ be a canonical scalar product. Note that one of these canonical objects uniquely determines the others. Everywhere below we fix on $\mathcal{H}$ a unique form $[x,y] = (Jx,y)$. At the same time let us note that in the question we consider, a concrete choice of Hilbert scalar product is not really essential. One needs only to fix the topology (defined by the above mentioned scalar product) and the structure of J.

Below non-negative (especially maximal non-negative) subspaces will play an important role. The set of all maximal non-negative subspaces of the Krein space $\mathcal{H}$ is denoted $\mathfrak{M}^+(\mathcal{H})$.

A subspace $\mathcal{L}$ is called *pseudo-regular* ([7]) if it can be presented in the form

$$\mathcal{L} = \hat{\mathcal{L}} \dot{+} \mathcal{L}_1, \tag{1.1}$$

where $\hat{\mathcal{L}}$ is a regular subspace and $\mathcal{L}_1$ is an isotropic part of $\mathcal{L}$ (i.e. $\mathcal{L}_1 = \mathcal{L} \cap \mathcal{L}^{[\perp]}$).

Proposition 1.1. ([3]) *Let:*

- $\mathcal{L}_+ \in \mathfrak{M}^+(\mathcal{H})$ *and be a pseudo-regular subspace;*
- $\mathcal{L}_1$ *be the isotropic subspace of* $\mathcal{L}_+$*;*
- $(\cdot,\cdot)'$ *be a scalar product on* $\mathcal{L}_1$*, such that the norm* $\sqrt{(x,x)'}$ *is equivalent to the original one;*
- $\mathcal{L}_- = \mathcal{L}_+^{[\perp]}$*;*

and let

$$\mathfrak{L}_+ = \hat{\mathfrak{L}}_+ \dot{+} \mathfrak{L}_1,\ \mathfrak{L}_- = \hat{\mathfrak{L}}_- \dot{+} \mathfrak{L}_1, \tag{1.2}$$

where $\hat{\mathfrak{L}}_+$ *and* $\hat{\mathfrak{L}}_-$ *are uniformly definite subspaces. Then one can define on* $\mathcal{H}$ *a canonical scalar product* $(\cdot,\cdot)$ *such that:*

$$\left.\begin{array}{lll} a)\ \text{on } \mathfrak{L}_1 & : & (\cdot,\cdot) \equiv (\cdot,\cdot)' \\ b)\ \mathfrak{L}_1 \perp \hat{\mathfrak{L}}_+ & , & \mathfrak{L}_1 \perp \hat{\mathfrak{L}}_- \\ c)\ \text{on } \hat{\mathfrak{L}}_+ & : & (\cdot,\cdot) = [\cdot,\cdot] \\ d)\ \text{on } \hat{\mathfrak{L}}_- & : & (\cdot,\cdot) = -[\cdot,\cdot] \end{array}\right\} \tag{1.3}$$

Define a special case of pseudo-regular subspaces: a non-negative (non-positive) subspace $\mathcal{L}$ is called *a subspace of the class* h^+ (h^-) if it is pseudo-regular and $\dim \mathcal{L}_1 < \infty$ for $\mathcal{L}_1$ as in (1.1). In Pontryagin spaces every subspace is pseudo-regular and every semi-definite subspace belongs to class h^+ or h^-.

Here the term "operator" means "bounded linear operator". By the symbol $B^{\#}$ we denote the operator J-adjoint (J-a.) to an operator B. For an operator A symbols: $\sigma(A)$ and $\sigma_p(A)$ mean respectively its *spectrum* and *point spectrum*. If $\lambda_0 \in \sigma_p(A)$ then the symbols $\mathfrak{N}_{\lambda_0}(A)$ and $\mathfrak{K}_{\lambda_0}(A)$ mean respectively the *root* linear manyfold (i.e. the set of all eigenvectors and root vectors) and the *eigenspace* of the operator A corresponding to the eigenvalue λ_0. If $\lambda_0 = 0$ then the subspace $\mathfrak{K}_{\lambda_0}(A)$ is also denoted $\operatorname{Ker} A$. Generally speaking $\mathfrak{N}_{\lambda_0}(A)$ can be a non-closed linear manyfold but for the type of A that we consider it is a subspace (i.e. a closed linear manyfold). For an operator A we set $\mathfrak{U}(A):\ = \cup_{\lambda\in\sigma_p(A)}\{\mathfrak{N}_\lambda(A)\}$ and $\mathfrak{U}_1(A):\ = \cup_{\lambda\in\sigma_p(A)}\{\mathfrak{K}_\lambda(A)\}$. In the same way for an operator family $\mathfrak{Y}$ we put $\mathfrak{I}(\mathfrak{Y}):\ = \cap_{A\in\mathfrak{Y}}\mathfrak{U}(A)$ and $\mathfrak{I}_1(\mathfrak{Y}):\ = \cap_{A\in\mathfrak{Y}}\mathfrak{U}_1(A)$. Now let a non-trivial $\mathfrak{Y}$-invariant linear manyfold $\mathcal{G} \subset \mathfrak{I}(\mathfrak{Y})$. Then for every $A \in \mathfrak{Y}$ there is a unique number $\lambda_{\mathcal{G}}(A)$ that is the eigenvalue of $A|_{\mathcal{G}}$, i.e.

$$\exists x \in \mathcal{G}:\ Ax = \lambda_{\mathcal{G}}(A)x,\ x \neq 0. \tag{1.4}$$

If an operator family $\mathfrak{Y}$ is such that the condition $A \in \mathfrak{Y}$ implies $A^{\#} \in \mathfrak{Y}$, then this family is said to be *J-symmetric*. Note that a group of J-unitary operators is J-symmetric. An operator algebra $\mathfrak{A}$ is said to be WJ^*-algebra if it is closed in the weak operator topology, J-symmetric and contains the identity I.

The symbol $\mathrm{Alg}\,\mathfrak{Y}$ means the minimal WJ^*-algebra which contains $\mathfrak{Y}$.

Definition 1.2. A J-symmetric operator family $\mathfrak{Y}$ belongs to the class D^+_κ if there is a subspace $\mathcal{L}_+$ in $\mathcal{H}$, such that

- $\mathcal{L}_+$ is $\mathfrak{Y}$-invariant,
- $\mathcal{L}_+ \in \mathfrak{M}^+(\mathcal{H}) \cap h^+$,
- $\dim(\mathcal{L}_+ \cap \mathcal{L}_+^{[\perp]}) = \kappa$.

Let $\Lambda = \{\lambda_k\}_1^n$ be a finite set of real numbers and let $\mathfrak{R}_\Lambda$ be the family $\{X\}$ of all Borel subsets of $\mathbb{R}$ such that $\partial X \cap \Lambda = \emptyset$, where ∂X is the boundary of X in $\mathbb{R}$. Let $E\colon\ X \mapsto E(X)$ be a countably additive (with respect to weak topology) function, that maps $\mathfrak{R}_\Lambda$ to a commutative algebra of projections in a Hilbert space $\mathcal{H}$, $E(\mathbb{R}) = I$. $E(X)$ is called *a spectral function* (*on* $\mathbb{R}$) *with the peculiar spectral set* Λ, the mention of Λ can be omitted. The symbol $\mathrm{Supp}(E)$ means the minimal closed subset S of $\mathbb{R}$, such that $E(X) = 0$ for every X: $X \subset \mathbb{R}\backslash S$ and $X \in \mathfrak{R}_\Lambda$. Besides the symbol E we shall use also as a notation of a spectral function the symbol E_λ, $\lambda \in \mathbb{R}$, where $E_\lambda = E((-\infty, \lambda))$. Note that the notion of peculiar set has no any direct connection with the behavior of spectral function and it means only that some points on $\mathbb{R}$ are distinguished. See below Definition 1.3 for some explanations.

In what follows the symbol let $\mathfrak{R}_\Lambda^{(0)}$ means the collection of all numerical subsets X such that $X \in \mathfrak{R}_\Lambda$ and $X \cap \Lambda = \emptyset$.

Definition 1.3. Let E be a spectral function with a peculiar spectral set Λ. If $\lambda \in \Lambda$ then λ will be called a peculiarity of E. Let λ be a peculiarity. Fix a set $X \in \mathfrak{R}_\Lambda$: $X \cap \Lambda = \{\lambda\}$. The peculiarity λ is called *regular* if the operator family $\{E(X \cap Y)\}_{Y \in \mathfrak{R}_\Lambda}$ is bounded, otherwise it is called *singular*.

A spectral function E that acts in a Krein space, is said to be *J-orthogonal* (J-orth.sp.f.) if $E(X)$ is a J-ortho-projection for every $X \in \mathfrak{R}_\Lambda$. The following theorem was announced in [11] and proved in [4].

Theorem 1.4. *Let $\mathfrak{Y} \in D_\kappa^+$ be a commutative family of J-s.a. operators with real spectra. Then there exists a J-orth.sp.f. E with a finite peculiar spectral set Λ (Λ may be the empty set), such that the* *following conditions hold*

$$\left.\begin{array}{ll}
a) & E_\lambda \in \operatorname{Alg}\mathfrak{Y} \text{ for all } \lambda \in \mathbb{R}\backslash\Lambda; \\
b) & \exists\ \mathfrak{Y}\text{-invariant } \mathcal{L}_+ \in h^+ :\ E(\Delta)\mathcal{H} = E(\Delta)\mathcal{L}_+[\dot{+}]E(\Delta)\mathcal{L}_-,\ \Delta \text{ being any closed segment satisfying } \Delta \in \mathfrak{R}_\Lambda^{(0)}; \\
c) & \forall\, A \in \mathfrak{Y},\ \exists \text{ a defined almost everywhere function } \phi(\lambda), \text{ such that for every interval } \Delta \in \mathfrak{R}_\Lambda^{(0)} \text{ the descomposition } AE(\Delta) = \int_\Delta \phi(\lambda)E(d\lambda) \text{ is valid;} \\
d) & \widetilde{\mathcal{H}} = \operatorname*{CLin}_{\Delta\in\mathfrak{R}_\Lambda^{(0)}}\{E(\Delta)\mathcal{H}\} \text{ is pseudo-regular and its isotropic part is finite dimensional;} \\
e) & \text{if } \mu \in \Lambda, \text{ then } \forall A \in \mathfrak{A} \text{ the set } \sigma(A|_{\mathcal{H}_\mu}), \text{ where } \mathcal{H}_\mu = \bigcap_{\mu\in\Delta} E(\Delta)\mathcal{H}, \text{ is a singletone } \{\lambda_A\}; \text{ moreover, there is a natural number } n \text{ (the same for all } A) \text{ such that } (A-\lambda_A I)^n\mathcal{H}_\mu = \{0\}; \\
f) & \text{if } \lambda_0 \in \Lambda, \text{ then either } \limsup_{\lambda\to\lambda_0}\|E_\lambda\| = \infty \text{ or at least for one } A \in \mathfrak{Y} \text{ the operator } A|_{\mathcal{H}_{\lambda_0}} \text{ isn't a sp. operator of scalar type.}
\end{array}\right\} \quad (1.5)$$

A spectral function E with a peculiar spectral set Λ satisfying Conditions (1.5) are called *an eigen spectral function* (e.s.f.) of the operator family $\mathfrak{Y}$.

It is evident that an e.s.f. E of an operator family $\mathfrak{Y} \in D_\kappa^+$ is not uniquely defined. At the same time there are some invariants related to E that depend of $\mathfrak{Y}$ only.

Proposition 2.1. *Let $E \in D_\kappa^+$ be a J-orthogonal spectral function with a peculiar spectral set Λ and let $\lambda_0 \in \Lambda$. The peculiarity λ_0 is singular if and only if the isotropic part the subspace*

$$\mathcal{H}_{\lambda_0} = \bigcap_{\Delta :\ \lambda_0 \in \Delta \in \mathfrak{R}_\Lambda} E(\Delta)\mathcal{H}$$

is non-trivial.

For the next step we need the following result from [3].

Proposition 2.2. *Let $A \in D_\kappa^+$ be a J-s.a. operator such that $\sigma(A) \subset \mathbb{R}$ and let $\mathcal{L}_+ \in h^+ \cap \mathfrak{M}^+(\mathcal{H})$ be A-invariant. Then there is on $\mathbb{R}$ the spectral function E_λ^A with a finite peculiar spectral set Λ, such that ($X \in \mathfrak{R}_\Lambda$)*

$$\left.\begin{array}{ll}
a) & E_\lambda \in \operatorname{Alg} A \text{ for every } \lambda \in \mathbb{R}\backslash\Lambda; \\
b) & \sigma(A|_{E(X)\mathcal{H}}) \subset \bar{X}; \\
c) & \text{if } X \cap \Lambda = \emptyset \text{ then the operator } AE(X) \text{ is a scalar spectral operator} \\
 & \text{and } AE(X) = \int_X \xi E(d\xi); \\
d) & \text{if } X \cap \Lambda \neq \emptyset \text{ then } AE(X) \text{ is not a scalar spectral operator;} \\
e) & \text{if } \Delta \in \mathfrak{R}_\Lambda \text{ and } \Delta \cap \Lambda = \emptyset \text{ then} \\
 & E(\Delta)\mathcal{H} = \mathcal{H}_\Delta^+[+]\mathcal{H}_\Delta^-,\ A\mathcal{H}_\Delta^+ \subset \mathcal{H}_\Delta^+, \\
 & A\mathcal{H}_\Delta^- \subset \mathcal{H}_\Delta^-,\ \mathcal{H}_\Delta^+ \text{ is uniformly positive} \\
 & \text{and } \mathcal{H}_\Delta^- \text{ is uniformly negative} \\
 & \text{(each of the subspaces } \mathcal{H}_\Delta^+ \text{ and } \mathcal{H}_\Delta^- \text{ can be equal to } \{0\}); \\
f) & \text{if } \Delta \in \mathfrak{R}_\Lambda \text{ and } \Delta \cap \Lambda \neq \emptyset \text{ then} \\
 & E(\Delta)\mathcal{L}_+ \cap \mathcal{L}_1 \neq \{0\}.
\end{array}\right\} \quad (2.1)$$

Remark 2.3. If $\sigma(A|_{E^A(X)\mathcal{H}}) \cap \sigma(A|_{\mathcal{L}_1}) = \emptyset$, then in Representation (2.1e) one can take $\mathcal{H}_X^+ = E^A(X)\mathcal{L}_+$ and $\mathcal{H}_X^- = E^A(X)\mathcal{L}_-$.

Proposition 2.4. *Let $A \in D_\kappa^+$ be a J-s.a. operator with $\sigma(A) \subset \mathbb{R}$, let E_λ^A be its e.s.f. and let $\lambda \in \sigma_p(A)$, $\widetilde{\mathcal{H}}_\lambda^A \colon = \cap_{\Delta\colon\ \lambda\in\Delta\in\mathfrak{R}_\Lambda} E^A(\Delta)\mathcal{H}$. Then dimension of the subspace $(A-\lambda I)\widetilde{\mathcal{H}}_\lambda^A$ does not exceed $3\kappa - 1$.*

Corollary 2.5. *Let $\mathfrak{Y} \in D_\kappa^+$ be a family of J-s.a. operators with real spectra and let a non-trivial $\mathfrak{Y}$-invariant linear manyfold $\mathcal{G} \subset \mathfrak{I}(\mathfrak{Y})$. Then its closure $\overline{\mathcal{G}}$ is also $\mathfrak{Y}$-invariant, $\overline{\mathcal{G}} \subset \mathfrak{I}(\mathfrak{Y})$ and there is a number m such that for every set of operators $A_1, A_2, \ldots, A_m \in \mathfrak{Y}$ the equality $(A_1 - \lambda_{\mathcal{G}}(A_1)I) \cdot (A_2 - \lambda_{\mathcal{G}}(A_2)I) \cdot \ldots \cdot (A_m - \lambda_{\mathcal{G}}(A_m)I)|_{\overline{\mathcal{G}}} = 0$ holds. Here $\lambda_{\mathcal{G}}(A_j)$ is defined by (1.4).*

Corollary 2.6. *Let $\mathfrak{Y} \in D_\kappa^+$ be a family of J-s.a. operators with real spectra and let a non-trivial $\mathfrak{Y}$-invariant subspace $\mathcal{G} \subset \mathfrak{I}(\mathfrak{Y})$. Then $\mathcal{K}_{\mathcal{G}}(\mathfrak{Y}) \colon = \cap_{A\in\mathfrak{Y}} \operatorname{Ker}((A - \lambda_{\mathcal{G}}(A)I)|_{\mathcal{G}}) \neq \{0\}$.*

Let us consider under the same hypothesis along with operators from $\mathfrak{Y}$ also operators from $\mathrm{Alg}(\mathfrak{Y})$. If a subspace $\mathcal{G} \subset \mathfrak{I}(\mathfrak{Y})$ is $\mathfrak{Y}$-invariant, then it is also $\mathrm{Alg}(\mathfrak{Y})$-invariant and due to Corollary 2.6 for every $A \in \mathrm{Alg}(\mathfrak{Y})$ there is (cf. (1.4)) a unique number $\lambda_{\mathcal{G}}(A)$ such that

$$A|_{\mathcal{K}_{\mathcal{G}}(\mathfrak{Y})} = \lambda_{\mathcal{G}}(A) I|_{\mathcal{K}_{\mathcal{G}}(\mathfrak{Y})}. \tag{2.2}$$

Proposition 2.7. *Let $\mathfrak{Y} \in D_\kappa^+$ be a family of J-s.a. operators with real spectra and let a non-trivial $\mathfrak{Y}$-invariant subspace $\mathcal{G}$ be such that $\mathcal{G} \subset \mathfrak{I}(\mathfrak{Y})$. Then (see* (2.2)*) there is a number m such that for every set of operators $A_1, A_2, \ldots, A_m \in \mathrm{Alg}(\mathfrak{Y})$ the equality $(A_1 - \lambda_{\mathcal{G}}(A_1)I) \cdot (A_2 - \lambda_{\mathcal{G}}(A_2)I) \cdot \ldots \cdot (A_m - \lambda_{\mathcal{G}}(A_m)I)|_{\mathcal{G}} = 0$ holds.*

Corollary 2.8. *Let $\mathfrak{Y} \in D_\kappa^+$ be a family of J-s.a. operators with real spectra. Then*

$$\mathfrak{I}(\mathfrak{Y}) = \mathfrak{I}(\mathrm{Alg}(\mathfrak{Y})).$$

Proof. Indeed,

$$\mathfrak{I}(\mathfrak{Y}) = \bigcup_{\upsilon \in \Upsilon} \mathcal{G}_\upsilon, \tag{2.3}$$

there Υ is an index set with finite or infinite cardinality and $\mathcal{G}_\upsilon$ is a $\mathfrak{Y}$-invariant subspace such that

$$\left.\begin{array}{ll} a) & \mathcal{G}_\upsilon \text{ is maximal in the following sense: if a linear manyfold } \widetilde{\mathcal{G}} \text{ is such that } \mathcal{G}_\upsilon \subset \widetilde{\mathcal{G}} \subset \mathfrak{I}(\mathfrak{Y}), \text{ then } \widetilde{\mathcal{G}} = \mathcal{G}_\upsilon, \\ b) & \text{for every } A \in \mathfrak{Y} \text{ the set } \sigma(A|_{\mathcal{G}_\upsilon}) \text{ is a singleton } \{\lambda_{\mathcal{G}_\upsilon}(A)\}, \end{array}\right\} \tag{2.4}$$

and we need to show that for every $A \in \mathrm{Alg}(\mathfrak{Y})$ and $\mathcal{G}_\upsilon$ there is a number m_υ such that $(A - \lambda_{\mathcal{G}_\upsilon}(A)I)^{m_\upsilon}|_{\mathcal{G}_\upsilon} = 0$, but this follows from Proposition 2.7. $\square$

Remark 2.9. The condition $\mathfrak{Y} \in D_\kappa^+$ in the assertions of Corollaries 2.5, 2.6, 2.8 and Proposition 2.7 are essential. In the example given below a commutative family $\mathfrak{Y}$ of J-s.a. nilpotent operators is such that $\mathfrak{I}(\mathfrak{Y}) = \mathcal{H}$ but $\mathfrak{I}_1(\mathfrak{Y}) = \{0\}$. Moreover $\mathrm{Alg}(\mathfrak{Y})$ contains no nilpotent but quasinilpotent operators, so $\mathfrak{I}(\mathrm{Alg}(\mathfrak{Y})) \neq \mathcal{H}$.

Example 2.10. Let $\{e_j\}_{j=-\infty}^{+\infty}$ be an orthonormalized basis in a Hilbert space $\mathcal{H}$, $Je_j = e_{-j}$ for all j. Set $A_1 e_{-1+3j} = e_{3j}$, $A_1 e_{3j} = e_{1+3j}$, $A_1 e_{1+3j} = 0$, $A_2 e_{-4+9j} = e_{-1+9j}$, ..., $A_2 e_{1+9j} = e_{4+9j}$, $A_2 e_{2+9j} = A_2 e_{3+9j} = A_2 e_{4+9j} = 0$, ..., $A_m e_{\left(3^m j - \sum_{p=0}^{m-1} 3^p\right)} = e_{\left(3^m j + 3^{m-1} - \sum_{p=0}^{m-1} 3^p\right)}$, ..., $A_m e_{\left(3^m j - 3^{m-1} + \sum_{p=0}^{m-1} 3^p\right)} = e_{\left(3^m j + \sum_{p=0}^{m-1} 3^p\right)}$, $A_m e_{\left(1 + 3^m j - 3^{m-1} + \sum_{p=0}^{m-1} 3^p\right)} = \dots = A_m e_{\left(3^m j + \sum_{p=0}^{m-1} 3^p\right)} = 0$, $j = \dots, -1, 0, 1, \dots$, $m = 1, 2, \dots$. It is easy to check that the family $\{A_m\}_1^\infty$ is J-symmetric and commutative, $A_m^3 = 0$ for every m but $\cap_{m=1}^\infty \operatorname{Ker}(A_m) = \{0\}$.

Proposition 2.11. *Let $\mathfrak{Y} \in D_\kappa^+$ be a family of J-s.a. operators with real spectra, let E_λ be its e.s.f. with a peculiar set Λ and let $\Xi(E)\colon = \{\xi\colon \xi \in \Lambda$ or $(E_{\xi+0} - E_\xi) \neq 0\}$. Then*

$$\mathfrak{I}(\mathfrak{Y}) = \bigcup_{\xi \in \Xi(E)} \mathcal{H}_\xi, \tag{2.5}$$

where $\mathcal{H}_\xi = \bigcap_{\epsilon>0} E([\xi - \epsilon, \xi + \epsilon])\mathcal{H}$.

Corollary 2.12. *If E is an e.s.f. of a family $\mathfrak{Y} \in D_\kappa^+$ of J-s.a. operators with real spectra, then the number of singular spectral peculiarities of E depends only of $\mathfrak{Y}$.*

Now let us go to a characterization of regular peculiarities.

Proposition 2.13. *Let $E \in D_\kappa^+$ be an e.s.f. of an operator family $\mathfrak{Y} \in D_\kappa^+$ and let $\mathcal{H}_\xi$ is defined by* (2.5)*. Then ξ is a regular peculiarity if and only if simultaneously*

- *$\mathcal{H}_\xi$ is a regular subspace;*
- *the subspace $\mathcal{H}_\xi \cap \mathfrak{I}_1(\mathfrak{Y})$ is degenerate.*

Let $\mathfrak{Y}$ be a commutative family of J-s.a. operators with real spectra. As a first step we consider the set $\mathfrak{I}_1(\mathfrak{Y}) \cap \mathfrak{I}_1(\mathfrak{Y})^{[\perp]}$. This set has the representation

$$\mathfrak{I}_1(\mathfrak{Y}) \cap \mathfrak{I}_1(\mathfrak{Y})^{[\perp]} = \bigcup_{\vartheta \in \Theta} \mathcal{Z}_\vartheta, \tag{2.6}$$

where Θ is an index set and $\mathcal{Z}_\vartheta$ is the isotropic part of the corresponding joint eigen-space for $\mathfrak{Y}$, i.e. for every $A \in \mathfrak{Y}$ there is a number $\lambda_{\mathcal{Z}_\vartheta}(A)$ such that $Ax = \lambda_{\mathcal{Z}_\vartheta}(A)x$ for all $x \in \mathcal{Z}_\vartheta = \big(\cap_{A\in\mathfrak{Y}} \operatorname{Ker}(A - \lambda_{\mathcal{Z}_\vartheta}(A)I)\big) \cap \big(\cap_{A\in\mathfrak{Y}} \operatorname{Ker}(A - \lambda_{\mathcal{Z}_\vartheta}(A)I)\big)^{[\perp]}$. Let

$$\mathfrak{P}_0(\mathfrak{Y}) \colon = \big\{\mathcal{Z}_\vartheta\big\}_{\vartheta\in\Theta}. \tag{2.7}$$

Next, for $\mathfrak{Y}$ we can consider Representation (2.3) that, evidently, can be find for every commutative operator family, but in the general case $\mathcal{G}_\upsilon$ is not a subspace but a linear manyfold. Due to the definition of $\mathcal{Z}_\vartheta$ for every $\theta \in \Theta$ there is the unique index $\upsilon_\vartheta \in \Upsilon$ such that $\mathcal{Z}_\vartheta \subset \mathcal{G}_{\upsilon_\vartheta}$. Let

$$\mathfrak{P}(\mathfrak{Y}) \colon = \big\{\mathcal{G}_{\upsilon_\vartheta}\big\}_{\vartheta\in\Theta}. \tag{2.8}$$

The next theorem is principal in this section and follows directly from Propositions 2.1 and 2.13.

Theorem 2.14. *Let $\mathfrak{Y} \in D_\kappa^+$ be a family of J-s.a. operators with real spectra, let E_λ be its e.s.f. with a peculiar set Λ. Then cardinalities of Λ and Θ from* (2.6) *coincide and*

$$\mathfrak{P}(\mathfrak{Y}) = \{\mathcal{H}_\lambda\}_{\lambda \in \Lambda}, \tag{2.9}$$

where $\mathfrak{P}(\mathfrak{Y})$ is defined by (2.8).

Corollary 2.15. *Let $\mathfrak{Y} \in D_\kappa^+$ be a family of J-s.a. operators with real spectra. Then both the number of singular peculiarities and the number of regular peculiarities are the same for all e.s.f. of $\mathfrak{Y}$.*

3.1. A particular case

In this subsection we consider a commutative operator family $\mathfrak{Y}$ of J-s.a. nilpotent operators. We also assume that $\mathfrak{Y}$ contains the infinite number of linearly independent members. Let us introduce a procedure that will be used for checking if $\mathfrak{Y} \in D_\kappa^+$.

Taking an arbitrary operator $A_1 \in \mathfrak{Y}$ such that $A_1 \neq 0$, we set $\mathcal{K}_1 \colon = \mathrm{Ker}(A_1)$. Since A_1 is nilpotent, $\mathcal{K}_1 \cap \mathcal{K}_1^{[\perp]} \neq \{0\}$. If for every $A \in \mathfrak{Y}$

$$A\mathcal{K}_1 \subset \mathcal{K}_1 \cap \mathcal{K}_1^{[\perp]},$$

the procedure is finished, otherwise we go to the next step, taking an arbitrary $A_2 \in \mathfrak{Y}$, such that $A_2\mathcal{K}_1 \not\subset \mathcal{K}_1 \cap \mathcal{K}_1^{[\perp]}$, and setting $\mathcal{K}_2 \colon = \mathcal{K}_1 \cap \mathrm{Ker}(A_2)$, etc. In general case, if for the step j the relation

$$A\mathcal{K}_j \subset \mathcal{K}_j \cap \mathcal{K}_j^{[\perp]} \tag{3.1}$$

holds for every $A \in \mathfrak{Y}$, this step is final, otherwise we fix some $A_{j+1} \in \mathfrak{Y}$ such that $A_{j+1}\mathcal{K}_j \not\subset \mathcal{K}_j \cap \mathcal{K}_j^{[\perp]}$, and set $\mathcal{K}_{j+1} \colon = \mathcal{K}_j \cap \mathrm{Ker}(A_{j+1})$. The constructed procedure will be called null-descended. It can contain finite or infinite number of steps. It is clear that the choice of the (finite or infinite) sequence A_1, A_2, $\ldots$ is ambiguous but this sequence uniquely defines the sequence $\mathcal{K}_1$, $\mathcal{K}_2$, $\ldots$.

Proposition 3.1. *If a commutative operator family $\mathfrak{Y}$ of J-s.a. nilpotent operators belongs to D_κ^+-class, then for every choice of A_1, A_2, ... the corresponding null-descended procedure contains a finite number of steps.*

Proof. Let us assume the contrary, i.e., that the null-descended procedure generates an infinite sequence $\{A_j\}_{j=1}^\infty$.

Now we set

$$\mathcal{L}_0 \colon = J\mathcal{L}_1,\ \mathcal{L}_2 \colon = \big(\mathcal{L}_0 \dot{+} \mathcal{L}_1\big)^{[\perp]} \cap \mathcal{L}_+,$$

$$\mathcal{L}_3 \colon = \big(\mathcal{L}_0 \dot{+} \mathcal{L}_1\big)^{[\perp]} \cap \mathcal{L}_1. \tag{3.2}$$

With no loss of generality one can assume (see Proposition 1.1) that the subspaces $\mathcal{L}_j$, $j = 0, 1, 2, 3$, are mutually orthogonal, $(\cdot,\cdot)|_{\mathcal{L}_2} = [\cdot,\cdot]|_{\mathcal{L}_2}$ and $(\cdot,\cdot)|_{\mathcal{L}_3} = -[\cdot,\cdot]|_{\mathcal{L}_3}$. Then the decomposition $\mathcal{H} = \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ yields the matrix representations

$$J = \begin{pmatrix} 0 & \tilde{V}^{-1} & 0 & 0 \\ \tilde{V} & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & -I_3 \end{pmatrix}, \tag{3.3}$$

$$A_j = \big(A_{pq}^{(j)}\big)_{p,q=0}^3, \tag{3.4}$$

where $A_{01}^{(j)} = A_{02}^{(j)} = A_{03}^{(j)} = A_{21}^{(j)} = A_{22}^{(j)} = A_{23}^{(j)} = A_{31}^{(j)} = A_{32}^{(j)} = A_{33}^{(j)} = 0$. Note that bloc-matrices

$$\begin{pmatrix} A_{00}^{(j)} & 0 \\ A_{10}^{(j)} & A_{11}^{(j)} \end{pmatrix}$$

act in a finite-dimensional vector space, so they also belong to a (different) real vector space and only a finite number of them are linearly independent. Let m_1 be a number such that every bloc-matrix with $j > m_1$ is a linear combination of first m_1 bloc-matrices, i.e.

$$\begin{pmatrix} A_{00}^{(j)} & 0 \\ A_{10}^{(j)} & A_{11}^{(j)} \end{pmatrix} = \sum_{l=1}^{m_1} \alpha_l^{(j)} \begin{pmatrix} A_{00}^{(l)} & 0 \\ A_{10}^{(l)} & A_{11}^{(l)} \end{pmatrix}, \quad \alpha_l^{(j)} = \overline{\alpha_l^{(j)}}.$$

Since $A_1|_{\mathcal{K}_{m_1}} = A_2|_{\mathcal{K}_{m_1}} = \ldots = A_{m_1}|_{\mathcal{K}_{m_1}} = 0$, the subspaces $\mathcal{K}_{m_1+1}$, $\mathcal{K}_{m_1+2}$, ... will be the same if we change A_{m_1+1}, A_{m_1+2}, ... for, respectively,

$$A_{m_1+1} - \sum_{l=1}^{m_1} \alpha_l^{(m_1+1)} A_l,\ A_{m_1+2} - \sum_{l=1}^{m_1} \alpha_l^{(m_1+2)} A_l,\ \ldots .$$

Thus, with no loss of generality we can assume that in Representation (3.4) the conditions

$$A_{00}^{(j)} = A_{10}^{(j)} = A_{11}^{(j)} = 0, \quad j \geq m_1 + 1$$

hold. Moreover, the subspace $\mathcal{L}_1$ is finite-dimensional, so there is a number m_2 such that for every $j > m_2$, vectors $x_+ \in \mathcal{L}_2$ and $x_- \in \mathcal{L}_3$ we have

$$A_j(x_+ + x_-) \in \widehat{\mathcal{L}} \colon = \operatorname*{Lin}_{\substack{y_+ \in \mathcal{L}_2,\, y_- \in \mathcal{L}_3 \\ l = m_1+1, \dots m_2}} \{A_l(y_+ + y_-)\}. \tag{3.5}$$

Let $\widehat{\mathcal{K}} \colon = \cap_{j=m_1+1}^{m_2} \operatorname{Ker}(A_j)$. Then $\widehat{\mathcal{K}} \subset \widehat{\mathcal{L}}^{[\perp]} = \big(\widehat{\mathcal{L}}^{[\perp]} \cap \mathcal{L}_0\big) \oplus \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$. If $y_+ \in \mathcal{L}_2$, $y_- \in \mathcal{L}_3$ and $x \in \big(\widehat{\mathcal{L}}^{[\perp]} \cap \mathcal{L}_0\big)$, then by (3.5) we have $[A_{m_2+1}x, y_+ + y_-] = [x, A_{m_2+1}(y_+ + y_-)] = 0$. Thus, $A_{m_2+1}x = 0$ and, therefore, $A_{m_2+1}\widehat{\mathcal{L}}^{[\perp]} \subset \widehat{\mathcal{L}}$. Since $\widehat{\mathcal{L}}$ is the isotropic part of $\widehat{\mathcal{L}}^{[\perp]}$, $\mathcal{K}_{m_2} \subset \widehat{\mathcal{K}} \subset \widehat{\mathcal{L}}^{[\perp]}$ and $A_{m_2+1}\mathcal{K}_{m_2} \subset \mathcal{K}_{m_2}$, the relation $A_{m_2+1}\mathcal{K}_{m_2} \subset \mathcal{K}_{m_2} \cap \mathcal{K}_{m_2}^{[\perp]}$ is now evident. The latter, nevertheless, contradicts to the hypothesis that no for one A_j Relation (3.1) holds. $\square$

Now let us consider the relation between the following conditions ($\mathfrak{Y}$ is a commutative nilpotent family of J-s.a. operators):

$$\mathfrak{Y} \in D_\kappa^+ \text{ for some } \kappa; \tag{3.6}$$

$$\left.\begin{array}{ll} a) & \text{for every } A \in \mathfrak{Y} \text{ the linear manyfold } A\mathcal{H} \text{ is finite-dimensional;} \\ b) & \text{for every realization of the null-descended procedure for } \mathfrak{Y} \text{ the number of steps is finite;} \end{array}\right\} \tag{3.7}$$

$$\left.\begin{array}{ll} a) & \text{for every } A \in \mathfrak{Y} \text{ the linear manyfold } A\mathcal{H} \text{ is finite-dimensional;} \\ b) & \text{for some realization of the null-descended procedure for } \mathfrak{Y} \text{ the number of steps is finite.} \end{array}\right\} \tag{3.8}$$

Theorem 3.2. *Conditions* (3.6), (3.7) *and* (3.8) *are equivalent.*

3.2. General case

Let $\mathcal{L}$ be a pseudo-regular $\mathfrak{Y}$-invariant subspace with finite-dimensional isotropic part and all operators $A|_{\mathcal{L}}$ are nilpotent. We need to adapt the definition of a null-descended procedure for the family $\mathfrak{Y}|_{\mathcal{L}}$ if the isotropic part of $\mathcal{L}$ is not trivial (if this part is trivial the procedure is practically the same as for the whole space). For arbitrary operator $A_1\colon\ A|_{\mathcal{L}} \neq 0$ we set $\mathcal{K}_1\colon = (A_1\mathcal{L})^{[\perp]} \cap \mathcal{L}$ and, in general, $\mathcal{K}_{j+1}\colon = \mathcal{K}_j \cap (A_{j+1}\mathcal{L})^{[\perp]}$, the stopping rule and selection of a sequence $A_1, A_2, \ldots$ are the same, i.e. they are related to Condition (3.1).

Theorem 3.3. *The following conditions are equivalent:*

$$\mathfrak{Y} \in D^{+}_{\kappa} \text{ in } \mathcal{L} \text{ for some } \kappa; \tag{3.9}$$

(3.10)

a) for every $A \in \mathfrak{Y}$ the linear manyfold $A\mathcal{L}$ is finite-dimensional;

b) for every realization of the null-descended procedure for $\mathfrak{Y}$ the number of steps is finite;

(3.11)

a) for every $A \in \mathfrak{Y}$ the linear manyfold $A\mathcal{L}$ is finite-dimensional;

b) for some realization of the null-descended procedure for $\mathfrak{Y}$ the number of steps is finite.

Next, we need the following result from [4] (an operator $U\colon \mathcal{L} \mapsto \mathcal{L}$ is said to be a $(\mathcal{L}, [\cdot,\cdot])$-unitary operator, if $U\mathcal{L} = \mathcal{L}$ and $[Ux, Uy] = [x, y]$ for every $x, y \in \mathcal{L}$):

Theorem 3.4. *Let $\mathcal{H}$ be a J-space and let $\mathfrak{W} = \{W\}$ be a commutative group of J-unitary operators. Then $\mathfrak{W} \in D_\kappa^+$ if and only if, there exists an $\mathfrak{W}$-invariant pseudo-regular subspace $\mathcal{L}$, such that:*

(i) *its isotropic part $\mathcal{L}_0 = \mathcal{L} \cap \mathcal{L}^{[\perp]}$ is a finite dimensional subspace;*

(ii) *$\mathfrak{W}_1 = \{W_1 = W|_{\mathcal{L}}\}_{W \in \mathfrak{W}}$ is a group of $(\mathcal{L}, [\cdot,\cdot])$-unitary operators belonging to D_κ^+;*

(iii) *for every $x, y \in \mathcal{L}^{[\perp]}$, the set*

$$\Omega_{x,y} = \{[Wx, y]\}_{W \in \mathfrak{W}}$$

is bounded.

Denote $\mathrm{Un}(\mathfrak{Y})$ the group of all J-unitary operators from $\mathrm{Alg}(\mathfrak{Y})$ and pass to the summarizing theorem.

Theorem 3.5. *Let $\mathfrak{Y}$ be a commutative family of J-s.a. operators with real spectra and let the set $\mathfrak{P}(\mathfrak{Y})$ be defined via (2.6), (2.7) and (2.8). Then $\mathfrak{Y} \in D_\kappa^+$ for some κ if and only if the following conditions hold:*

- *the cardinality of Θ is finite;*
- *all elements of $\mathfrak{P}(\mathfrak{Y})$ are regular or pseudo-regular;*
- *if $\mathcal{G}_{\upsilon_\vartheta}$ is pseudo-regular, then its isotropic part is finite-dimensional;*
- *for every $\vartheta \in \Theta$ and $A \in \mathfrak{Y}$ the linear manyfold $\big(A - \lambda_{\mathcal{Z}_\vartheta}(A)I\big)\mathcal{G}_{\upsilon_\vartheta}$ is finite-dimensional;*
- *for every $\vartheta \in \Theta$ and some (every) realization of the null-descended procedure for the family $\Big\{\big(A - \lambda_{\mathcal{Z}_\vartheta}(A)I\big)|_{\mathcal{G}_{\upsilon_\vartheta}}\Big\}_{A\in\mathfrak{Y}}$ the number of steps is finite.*
- *for every $x, y \in \mathfrak{P}(\mathfrak{Y})^{[\perp]}$ the set $\big\{[Ux, y]\big\}_{U\in\mathrm{Un}(\mathfrak{Y})}$ is bounded.*

4. Closing remarks

J-unitary operators with invariant subspaces of the type h^+ were appeared firstly in the Helton′s paper [8] and a successive development of this direction (covering so-called H and $K(H)$ classes) was given by Azizov (see [2] for details). The D_κ^+-class was introduced by Strauss [10]. A comparative analysis of different classes of J-s.a. operators in Krein spaces (including D_κ^+-class) generating some kinds of spectral resolutions can be found in [5]. Let us note also that some of results of the presented paper, for instance, Corollary 2.5, are well known for the case of individual operators (see [2], § III.5).

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