Rows versus Columns

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reporting on the joint effort with Manfred Möller (WITS, SA)

The anatomy of matrices of unbounded operators will be presented in some detail.

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On the other hand, the odd examples are unsurpassed as they are settled within the richest structure; the most exhaustive situation to happen.

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This is a matrix operator

$$\mathscr{D}(\boldsymbol{A}) \stackrel{\text{def}}{=} (\mathscr{D}(A) \cap \mathscr{D}(C)) \oplus (\mathscr{D}(B) \cap \mathscr{D}(D))$$

$$\boldsymbol{A}(f \oplus g) \stackrel{\text{def}}{=} (Af + Bg) \oplus (Cf + Dg), \quad f, g \in \mathscr{D}(\boldsymbol{A})$$

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$$\boldsymbol{R}_{(R_1,R_2)}(f_1 \oplus f_2) \stackrel{\text{def}}{=} R_1 f_1 + R_2 f_2, \quad f_1 \oplus f_2 \in \mathscr{D}(\boldsymbol{R}_{(R_1,R_2)}).$$

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$$\mathscr{D}(\boldsymbol{C}_{(C_1,C_2)}) \stackrel{\text{def}}{=} \mathscr{D}(C_1) \cap \mathscr{D}(C_2),$$

$$\boldsymbol{C}_{(R_1,R_2)} f \stackrel{\text{def}}{=} C_1 f_1 \oplus C_2 f_2, \quad f \in \mathscr{D}(\boldsymbol{C}_{(R_1,R_2)}).$$

The anatomy

This is a rather formal (and elementary) linear algebra

The subject for dissection

Let

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Then

$$\boldsymbol{A} = \boldsymbol{C}_{(\boldsymbol{R}_{(A_{11},A_{21})},\boldsymbol{R}_{(A_{12},A_{22})})}$$

The lesson

The goal

Examine closures and adjoints (duals, remember: it has to serve other than Hilbert spaces as well) of matrix operators.

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Examine those of row and column operators!

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The case when at most of one of the operators in a row or/and in a column is bounded is reluctantly not excluded (this is just trivial; everything goes as someone might have dreamed of).

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About ‘crossing’

By $\times$ we denote always the adjoint operation applied entrywise regardless an object in question is: a row, a column or a matrix.

Row and column operators

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To recompase you the above

If $\mathscr{R}(R_2) \subset \mathscr{R}(R_1)$, R_1 is injective and $R_1 R_2^{-1}$ extends to a bounded operator K, say, then

$$\overline{\boldsymbol{R}_{(R_1, R_2)}} = \boldsymbol{R}_{(\overline{R_1}, 0)} \begin{pmatrix} I & K \\ 0 & I \end{pmatrix}.$$

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In other words, the entrywise closure does the job!

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Be in ℓ^2 space with the usual '$0-1$' basis $(e_n)_n$, please. Set

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$$\mathscr{D}(R_1) \stackrel{\text{def}}{=} \left\{ f = \sum_{k=1}^{\infty} \gamma_k e_k : \sum_{k=1}^{\infty} |\gamma_k|^2 < \infty, \ \sum_{k=1}^{\infty} |\gamma_k \beta_k|^2 < \infty \right\},$$

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Then the patient calculation makes it.

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The troublemaker shows its real face

Let C_1 and T be densely defined operators such that

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Then, after $C_2 \stackrel{\text{def}}{=} T - C_1$

$$\mathscr{D}(\boldsymbol{C}^{\times}_{(C_1,C_2)}) \subsetneq \mathscr{D}(\boldsymbol{C}^{*}_{(C_1,C_2)}).$$

A little something again

Suppose $\mathscr{D}(C_1) = \mathscr{D}(C_2)$ and C_1 is injective with $C_2 C_1^{-1}$ having a bounded extension. Let $\mathscr{D}_0$ be a subspace of $\mathscr{D}(\boldsymbol{C}^*_{(C_1,C_2)})$ such that there exists (at least one) $v_0 \in \mathscr{H}_2 \setminus \{0\}$ for which $(-Kv_0, v_0) \in \mathscr{D}_0$.

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Then $\mathscr{D}_0$ is a core of $\boldsymbol{C}_{(C_1,C_2)}$ if and only if $\mathscr{D}_0$ is dense in $\mathscr{H}_1 \oplus \mathscr{H}_2$ and $\boldsymbol{C}_{(I,K^*)}(\mathscr{D}_0) \cap \mathscr{D}(C_1^*)$ is a core for C_1^*.

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2° $\overline{\boldsymbol{C}^{\times}_{(C_1,C_2)}} = \boldsymbol{C}^{*}_{(C_1,C_2)}$ if and only if $\overline{\boldsymbol{C}_{(C_1,C_2)}} = \boldsymbol{C}_{(\bar{C}_1,\bar{C}_2)}$.

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1^{o} $\boldsymbol{C}_{(C_1,C_2)}$ is closed if so are C_1 and C_2;

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Again some clouds are coming (next slide please).

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An eccentric example

Let $\boldsymbol{C}$ be a merely unbounded operator which is closable. Then $\boldsymbol{C}$ decomposes as $\boldsymbol{C}_{(C_1,C_2)}$ with C_1 not closable.

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Hint: take $x \notin \mathscr{D}(\boldsymbol{C}^*)$ and set $C_1 \stackrel{\text{def}}{=} P\boldsymbol{C}$ and $C_2 \stackrel{\text{def}}{=} (I-P)\boldsymbol{C}$ where P is the rank one projection on $\{x\}$.

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and notice the difference! The latter is like coupling, isn't it?

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$$\boldsymbol{A} = \boldsymbol{C}_{(\boldsymbol{R}_{(A_{11},A_{21})},\boldsymbol{R}_{(A_{12},A_{22})})}$$

Applying all the permissible procedures which help one gets

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1^{o} $\boldsymbol{A}^{\times}$ is closable;

2^{o} if $\boldsymbol{A}^{\times}$ is densely defined, $\boldsymbol{A}$ is closable;

3^{o} even if all A_{ij} are closed, $\boldsymbol{A}$ may not be closable (this comes out from combining previous examples).

$\mathscr{H}_1 = \mathscr{H}_2 = \mathscr{L}^2(0,1)$ and the operators A_{ij} are defined by

$$\begin{aligned}
&\mathscr{D}(A_{11}) = \{f \in W_2^1(0,1) : f(0) = 0\},\ A_{11}f = f',\\
&\mathscr{D}(A_{12}) = L_2(0,1),\ A_{12} = 0,\\
&\mathscr{D}(A_{21}) = \{f \in W_2^1(0,1) : f(1) = 0\},\ A_{21}f = f',\\
&\qquad A_{22} = -A_{21},
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where $W_2^1(0,1)$ denotes the usual Sobolev space of order 1.

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The worst example for today

Then all A_{ij} are closed, $\boldsymbol{A}$ as well as $\boldsymbol{A}^{\times}$ is densely defined but

$$\overline{\boldsymbol{A}^{\times}} \subsetneq \boldsymbol{A}'.$$