Normal and Hyponormal Matrices in Inner Product Spaces

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joint work with Christian Mehl

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Introduction

On $\mathbb{C}^n$ consider for a Hermitian matrix H

$$[x, y] := (Hx, y)_{\mathbb{C}^n} .$$

Let A be a matrix.
H regular: Define $A^{[*]}$ via

$$[x, Ay] = \left[A^{[*]}x, y\right]$$

Define H-selfadjoint ($A^{[*]} = A$) H-unitary ($A^{[*]} = A^{-1}$) and H-normal ($A^{[*]}A = AA^{[*]}$) as usual.

H singular: ??
In this case one defines H-selfadjoint and H-unitary via

$$A^*H = HA \quad \text{resp.} \quad A^*HA = H. \qquad (1)$$

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H-normal Matrices

Question: How to define H-normal matrices if H is singular?

In [Li,Tsing,Uhlig '96], [Mehl,Ran,Rodman '04] via the Moore-Penrose inverse $H^\dagger$ of H:

$$HAH^\dagger A^* H = A^* HA \tag{2}$$

Definition

We call a matrix A satisfying (2) *Moore-Penrose H-normal*.

Our approach:

1. Define the H-adjoint in the sense of linear relations.
2. Define then H-normal matrices.

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Linear relations

Assume always: H singular Hermitian and A is a matrix.

$$[x,y] := (Hx,y)_{\mathbb{C}^n}\,.$$

Identify $A \longleftrightarrow \operatorname{graph} A$ Then A is a linear relation.

$$\operatorname{dom} A = \left\{ x \ : \ \begin{pmatrix} x \\ y \end{pmatrix} \in A \right\},$$ the domain of A,

$$\operatorname{mul} A = \left\{ y \ : \ \begin{pmatrix} 0 \\ y \end{pmatrix} \in A \right\},$$ the multivalued part of A,

$$A^{-1} = \left\{ \begin{pmatrix} y \\ x \end{pmatrix} \ : \ \begin{pmatrix} x \\ y \end{pmatrix} \in A \right\},$$ the inverse of A

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The linear relation $A^{[*]}$ is called the *H-adjoint* of A,

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Lemma

1. *We have $A^{[*]} = H^{-1} A^* H$ (similar to the case H regular).*
2. *We have $\operatorname{mul} A^{[*]} = \ker H$.*
3. *Hence $A^{[*]}$ is a matrix if and only if H is regular.*

Example

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\operatorname{dom} A^{[*]} = \{0\} \times \mathbb{C}, \qquad \operatorname{mul} A^{[*]} = \{0\} \times \mathbb{C}.$$

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H-symmetric and H-isometric matrices

Definition

1. We call A H-*symmetric* if $A \subset A^{[*]}$.
2. We call A H-*isometric* if $A^{-1} \subset A^{[*]}$.

First task: Compare this with (1), i.e.

$$A^* H = HA \quad \text{resp.} \quad A^* HA = H.$$

Choosing a basis of $\mathbb{C}^n$, we may always assume that H and A have the forms, with H_1 is regular,

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}. \tag{3}$$

Lemma

We have $\operatorname{dom} A^{[*]} = \mathbb{C}^n \iff \operatorname{ran}(A^* H) \subseteq \operatorname{ran} H \iff A_2 = 0.$

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Write A and H in the form

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}.$$

Theorem

The following statements are equivalent.

(i) A is H-symmetric, $A \subset A^{[*]}$.

(ii) $A^* H = HA$ (that is (1)).

(iii) A_1 is H_1-selfadjoint and $A_2 = 0$,

(iv) $A^{[*]} = (A^{[*]})^{[*]}$ (i.e. the relation $A^{[*]}$ is H-selfadjoint).

(v) $A^{[*]} = A + (\ker H \times \ker H)$.

In particular, $\ker H$ is A-invariant.

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Definition

A matrix A in $\mathbb{C}^n$ is called H-normal if $AA^{[*]} \subseteq A^{[*]}A$.

Rem.: Other inclusion does not include all H-sym. matrices.

Theorem

A is H-normal $\Longleftrightarrow$ A_1 is H_1-normal and $A_2 = 0$ $\Longleftrightarrow$
A is Moore-Penrose H-normal and $A^{[*]}(A^{[*]})^{[*]} = (A^{[*]})^{[*]}A^{[*]}$.
Moreover, $\ker H$ is A-invariant.

Remarks:

- H-symmetric and H-isometric matrices are H-normal.
- [Mehl, Ran, Rodman '04]: Exists Moore-Penrose H-normal matrices A such that $\ker H$ is not A-invariant.
- H-normal matrices is a strict subset of the set of Moore-Penrose H-normal matrices.

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With [L '71] it is easy to see:

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Let A be H-normal and $\mathcal{M}_0$ be an H-nonnegative A-invariant subspace that is also invariant for $A^{[*]}$. Then there exists an A- and $A^{[*]}$-invariant maximal H-nonnegative subspace $\mathcal{M}$ with

$$\mathcal{M}_0 \subseteq \mathcal{M}.$$

Next Step: Drop invariance with respect to $A^{[*]}$.

In [Mehl, Ran, Rodman '04]: Exists H-normal matrix (even if H is regular) with invariant H-nonnegative subspace that cannot be extended to an invariant maximal H-nonnegative subspace. Thus, stronger conditions.

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Invariant subspaces II

Theorem

Let A be H-normal and let $\mathcal{M}_0$ be an H-positive A-invariant subspace. Let $\mathcal{M}_{com}$ such that $\mathcal{M}_0^{[\perp]} = \mathcal{M}_{com} \dot{+} \ker H$. Define

$$X := PA|_{\mathcal{M}_{com}} : \mathcal{M}_{com} \to \mathcal{M}_{com},$$

where P is the projection onto $\mathcal{M}_{com}$ along $\mathcal{M}_0 \dot{+} \ker H$. Assume that

$$\sigma(X + X^{[*]}) \subseteq \mathbb{R} \quad \text{or} \quad \sigma(X - X^{[*]}) \subseteq i\mathbb{R}.$$

Then there exists an A-invariant maximal H-nonnegative subspace $\mathcal{M}$ that contains $\mathcal{M}_0$ and that is also $A^{[*]}$-invariant.

Hyponormal matrices

In a definition the foolowing line should appear:

$$A^{[*]}A - AA^{[*]} \text{ is } H\text{-nonpositive.} \tag{4}$$

Recall: A linear relation B is H*-nonpositive* if B is H-symmetric and

$$[y, x] \leq 0 \quad \text{for all} \begin{pmatrix} x \\ y \end{pmatrix} \in B.$$

Problem: . Let H_1 and A_2 regular $n \times n$ matrices,

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}.$$

Then $\mathrm{dom}\ (AA^{[*]}) \cap \mathrm{dom}\ (A^{[*]}A) = \{0\}$ and (4) is always true. ($\longrightarrow$ unbounded operators) Remember:

Lemma

We have $\mathrm{dom}\, A^{[*]} = \mathbb{C}^n \iff \mathrm{ran}\,(A^*H) \subseteq \mathrm{ran}\, H \iff A_2 = 0.$

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Let A be a matrix as above. The following are equivalent.

(i) *The domain of the linear relation $A^{[*]}A$ is $\mathbb{C}^n$.*

(ii) $A_2^* H_1 A_1 = 0$ *and* $A_2^* H_1 A_2 = 0$.

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A linear relation A is called *strongly H-hyponormal* if A is H-hyponormal and if $(A^{[*]})^n A^n$ has full domain for all $n \in \mathbb{N}$.

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