On eigenvalue bounds for indefinite selfadjoint operators.

Krešimir Veselić, Fernuniversität Hagen

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Let H be selfadjoint and A symmetric. What can be said on the discrete eigenvalues λ_k, μ_k of H, $T = H + A$, respectively?

Depends on how the sum $H + A$ is defined. Known posibilities are

$$\|A\psi\|^2 \le a\|\psi\|^2 + b\|H\psi\|^2, \; b < 1,$$

$$|(A\psi, \psi)| \le a\|\psi\|^2 + b(H\psi, \psi), \; b < 1,$$

$$|(A\psi, \psi)| \le a\|\psi\|^2 + b(|H|\psi, \psi), \; b < 1, \quad (1)$$

— the most general case. Expected bound:

$$|\mu_k - \lambda_k| \le a + b|\lambda_k|$$

Is this true? When exactly (spectral gaps, too)? How to compute $|H|$?

All our results are formulated more generally — for quadratic form sums (instead of $(A\psi, \psi)$ we have just a form $\alpha(\psi, \psi)$). We stick to the operator symbols for notational simplicity.

Today's topics

- A general eigenvalue bound, covering known results for finite matrices, uses no variational principles, valid whenever feasible. Spectral gaps included.

- 'Krein space connection': typical selfadjoint indefinite operator is often given as $T = GJG^*$ where all factors have bounded inverses. If the associated J-selfadjoint operator

$$S = JG^*G$$

is regular at infinity then it yields a new attractive formula for $|T|$.

- Some new regularity criteria for such S are derived.

- Some now types of operator matrices are considered.

The pseudo-Friedrichs construction (Kato):

$$|(A\psi,\psi)| \leq \|H_1\psi\|^2,\ H_1 = a + b|H|,\ b < 1.$$

$$T = H_1^{-1/2}(\mathrm{sign}\, H + C)H_1^{-1/2},$$

$$(C\psi,\phi) = (AH_1^{-1/2}\psi, H_1^{-1/2}\phi),\ \|C\| \leq 1.$$

Theorem 1 *The form bound above implies*

$$|\mu_k - \lambda_k| \leq a + b|\lambda_k| \tag{2}$$

in any 'firewalled' interval $\mathcal{W}$ such that $\mathcal{W} \cup \sigma_{ess}(H) = \emptyset$. Here, depending on $\mathcal{W}$, both μ_k and λ_k are either increasingly or decreasingly ordered

A 'firewall' is a point d which is not crossed by the spectral points of several analytic operator families used in the proof, in particular of $T_\epsilon = H + \epsilon A$. Possible $\mathcal{W}$'s are

$$(d_1, d_2), \quad (d,\infty), \quad (-\infty, d), \quad (-\infty,\infty).$$

An 'optimally placed' firewall d in a maximal resolvent interval λ_+, λ_- of H is

$$d = \begin{cases} \frac{2b\lambda_+\lambda_- + a(\lambda_+ + \lambda_-)}{b(\lambda_+ + \lambda_-) + 2a}, & \lambda_- \geq 0 \\ \\ \frac{a(\lambda_+ + \lambda_-)}{b(\lambda_+ - \lambda_-) + 2a}, & \lambda_- \leq 0 \end{cases} \qquad (3)$$

if the constants a, b satisfy

$$\begin{aligned} &\frac{b(\lambda_+ + \lambda_-) + 2a}{\lambda_+ - \lambda_-} < 1, \quad \lambda_- \geq 0 \\ &b + \frac{2a}{\lambda_+ - \lambda_-} < 1, \qquad \lambda_- \leq 0 \end{aligned} \qquad (4)$$

Here, for definiteness, $\lambda_+ > 0$. The proof of the theorem uses the analyticity of the operator families

$$T_\epsilon = H + \epsilon A, \quad \tilde{T}_\epsilon = (1-\epsilon)T + \epsilon(H + a + b|H|)$$

$$-1 \leq \epsilon \leq 1,$$

Now the monotonicity of $\tilde{T}_\epsilon$ carries over to the eigenvalues

$$\lambda_k - a - b|\lambda_k| \leq \mu_k \leq \lambda_k + a + b|\lambda_k|.$$

In fact, the full strength of the obtained estimate is

$$\min \sigma(C)(a+b|\lambda_k|) \le \mu_k - \lambda_k \le \max \sigma(C)(a+b|\lambda_k|)$$

with C from

$$T = H_1^{-1/2}(\operatorname{sign} H + C)H_1^{-1/2}.$$

For $b = 0$, $a = \|A\|$ this recovers the known sharp estimate

$$\min \sigma(A) \le \mu_k - \lambda_k \le \max \sigma(A)$$

for the eigenvalues of $T = H + A$.

Typical way of constructing an unbounded indefinite selfadjoint T with a spectral gap at zero is a factorisation:

$$T = GJG^*.$$

E.g. the pseudo-Friedrichs construction for "$T = H + A$" reads

$$T = |H|^{-1/2}(\operatorname{sign} H + C)|H|^{-1/2},$$

$$(C\psi, \phi) = (A|H|^{-1/2}\psi, |H|^{-1/2}\phi), \ \|C\| \le 1.$$

The associated operator

$$S = JG^*G$$

is J-selfadjoint and has the same spectrum as T. Moreover, S is *J-positive definite* (take for simplicity J as a symmetry: $J = J^{-1}$).

A closer connection between T and S holds, if S is 'regular at infinity', that is,

$$S_0 = F^{-1}SF, \quad \text{with } J\text{-unitary } F$$

is selfadjoint i.e. commutes with J (*J-unitary diagonalisability*).

Theorem 2 *Let* G^{-1} *be bounded,*

$$T = GJG^*,\ J = J^* = J^{-1}$$

and let S *be* J*-diagonalisable:*

$$S_0 = F^{-1}SF, \quad S_0 J = JS_0,\ F\ J\text{-unitary.}$$

Then $JS_0 = |S_0|$ *is positive definite and*

$$U^{-1}TU = S_0 \tag{5}$$

where $U = GF(JS_0)^{-1/2}$ *is unitary. Moreover,*

$$|T| = \sqrt{T^2} = GFF^*G^* \tag{6}$$

and

$$\mathcal{D}(G^*) = \mathcal{D}(|T|^{1/2}). \tag{7}$$

Conversely, (7) implies the J*-unitary diagonalisability of* S*.*

Application: for $S = JP$, P pos. definite, take $G = P^{1/2}$, $T = P^{1/2}JP^{1/2}$.

S regular at ∞ $\Leftrightarrow$ $\mathcal{D}(P^{1/2}) = \mathcal{D}(|T|^{1/2})$.

A more practical sufficient regularity criterion:

Theorem 3 *Let S be such that $P = JS$ is selfadjoint and positive definite for some $J = J^* = J^{-1}$. Let Θ be positive definite and such that its form domain coincides with that of P and $\Theta J = J\Theta$. Then S is J-diagonalisable. All diagonalising F have the same norm and*

$$\|F\|^2 = \kappa(F) \leq \kappa(P^{1/2}\Theta^{-1/2}), \tag{8}$$

for any such Θ. Here $\kappa(A) = \|A\|\|A^{-1}\|$ (the condition number).

Q: Is this condition necessary?

Sketch of the proof of Th. 3. Step 1: $S = JP$ is bounded. Then the result is known to hold (K. Veselić and I. Slapničar 2000). Here the regularity itself is trivial, in fact F can be taken as

$$F = (J\,\mathsf{sign}(S))^{1/2},$$

where $J\,\mathsf{sign}(S)$ is selfadjoint and positive definite. The uniform bound

$$\|F\|^2 = \kappa(F) \leq \kappa(P^{1/2}\Theta^{-1/2}),$$

is nontrivial and quite technical.

Step 2: S is unbounded. By assumption

$$\Pi = (P^{1/2}\Theta^{-1/2})^* P^{1/2}\Theta^{-1/2}$$

is bicontinuous. Approximate by cut-off: Take $d > 0$ and

$$f_d(t) = \begin{cases} t, & t \leq d, \\ d, & t > d, \end{cases}$$

$$\Theta_d = f_d(\Theta),\ P_d = \Theta_d^{1/2}\Pi\Theta_d^{1/2},\ S_d = JP_d$$

The (generalised) strong convegence

$$S_d \to S,\ d \to \infty$$

and the uniform bound

$$\kappa(P_d^{1/2}\Theta_d^{-1/2}) \le \kappa(P^{1/2}\Theta^{-1/2})$$

allow the use of the classical approximation result by G. Bade (1954) which implies the strong convergence

$$F_d = (J\,\mathsf{sign}(S_d))^{1/2} \to$$

$$F = (J\,\mathsf{sign}(S))^{1/2},\ d \to \infty.$$

Here F is non-negative; as a product of two reflections it is positive definite, J-unitary and $F^{-1}SF$ commutes with J (again by strong convergence). Q.E.D.

Nice: The condition number above — a measure of closeness of two equivalent topologies — gives a bound for the spectral measure of a J-positive operator S.

Example:

$$S\psi(x) = -\operatorname{sign} x \frac{d^2}{dx^2}\psi(x), \ \psi(-1) = \psi(1) = 0.$$

This is known to be regular (Najman, Čurgus).

Does this S satisfy our sufficiency criterion above? NO!

What is $\|F\|$ here?

Numerical experiment: n-point discretisation:

n	cond(F)	eigenvalues...				
50	3.5	21.4	115.5	283.3	521.9	827
100	3.7	21.9	118.3	291.7	541.1	865
500	4.2	22.2	120.4	297.3	552.8	886
1000	4.4	22.3	120.6	297.9	554.0	888
2000	4.5	22.3	120.7	298.2	554.6	889

How to find a Θ for $S = JP$?

We call P *J-strongly diagonally dominant*, if

$$J\mathcal{D}(P^{1/2}) \subseteq \mathcal{D}(P^{1/2})$$

and

$$|(P^{1/2}Q_+\psi, P^{1/2}Q_-\phi)| \ \le$$

$$\beta\|(P^{1/2}Q_+\psi\|\|, P^{1/2}Q_-\phi)\|, \ \psi, \phi \in \mathcal{D}(P^{1/2})$$

where

$$Q_\pm = (J \pm 1)/2, \text{ and } \beta < 1.$$

In this case take Θ as the diagonal block of P and obtain

$$\|F\|^2 \le \sqrt{\frac{1+\beta}{1-\beta}}.$$

Multiplicative perturbations.

The operator $T = GJG^*$ with $J = J^* = J^{-1}$ and G^{-1} bounded, is perturbed into

$$\tilde{T} = \tilde{G}JG^*, \ \tilde{G} = G + \delta G$$

and either

$$\|\delta G^* G^{-*}\| \le \beta < 1 \text{ or } \|\delta G G^{-1}\| \le \beta < 1$$

Consider the first case (the second case is very different):

$$\tilde{G}^* = (1 + \delta G^* G^{-*})G^*.$$

So, $\tilde{T}$ automatically selfadjoint.

Theorem 4 *The eigenvalues of the operators $\tilde{T}$, T, respectively, are bounded as*

$$|\tilde{\mu}_k - \mu_k| \le b|\mu_k|.$$

in any window, provided that

$$b = (2\beta + \beta^2)\|F\|^2 < 1.$$

An application to $S\psi(x) = -\operatorname{sign} x \, \frac{d^2}{dx^2}\psi(x)$.

We perturb $\operatorname{sign} x$ into $(1 + \rho(x)) \operatorname{sign} x$, $|\rho(x)|$ small. This leads to

$$|\tilde{\mu}_k - \mu_k| \leq (2\beta + \beta^2)\|F\|^2 |\mu_k|.$$

with $\beta = \|\rho\|_\infty$.

How realistic is this bound?

Application to the 'quasidefinite' operator matrix

$$T = \begin{bmatrix} A & B \\ B^* & -C \end{bmatrix} \tag{9}$$

A, C are symmetric non-negative. The main special cases:

1. A, C are dominant, i.e. positive definite and
$$\|A^{-1/2}BC^{-1/2}\| < \infty$$

2. B is dominant, i.e. bicontinuous and
$$\|(BB^*)^{-1/4}A(BB^*)^{-1/4}\| < \infty,$$
$$\|(B^*B)^{-1/4}C(B^*B)^{-1/4}\| < \infty.$$

   In both cases the matrix defines a self-adjoint operator. Applications to Stokes and Dirac operators.

In the first case

$$T = \begin{bmatrix} A^{1/2} & 0 \\ 0 & C^{1/2} \end{bmatrix} \begin{bmatrix} 1 & W \\ W^* & -1 \end{bmatrix} \begin{bmatrix} A^{1/2} & 0 \\ 0 & C^{1/2} \end{bmatrix}$$

with $W = A^{-1/2} B C^{-1/2}$. 'Relative bound' $\|W\|$ is finite, but need not be < 1 !

Furthermore,

$$T = GJG^*,$$

$$G = \begin{bmatrix} A^{1/2} & 0 \\ 0 & C^{1/2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ W^* & (1 + W^*W)^{1/2} \end{bmatrix},$$

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Theorem 5 *If, in addition,*

$$\gamma = \|B^* A^{-1}\| < \infty$$

then

$$\|F\|^2 \leq \frac{1}{2}(2 + \gamma^2 + \gamma\sqrt{\gamma^2 + 4}).$$