Kato decompositions for quasi - Fredholm relations

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Range space relations

$(\mathfrak{H},\ (\cdot,\cdot))$: Hilbert space

$\mathfrak{M} \subset \mathfrak{H}$ range subspace of $\mathfrak{H}$ if

$\mathfrak{M} = \operatorname{ran} A$ for some $A \in \mathbf{L}(\mathfrak{K}, \mathfrak{H})$

$\exists$ inner product $(\cdot,\cdot)_+$ on $\mathfrak{M}$

$\|u\|_+ \geq c\|u\|_{\mathfrak{H}}$, $u \in \mathfrak{M}$, $c > 0$

$(\mathfrak{M},\ (\cdot,\cdot)_+)$ Hilbert space

Note: If $\mathfrak{M}$ closed then $\mathfrak{M}$ range space

Neubauer's Lemma

$\mathfrak{M}$, $\mathfrak{N}$ range subspaces of $\mathfrak{H}$

If $\mathfrak{M} + \mathfrak{N}$, $\mathfrak{M} \cap \mathfrak{N}$ closed then $\mathfrak{M}$, $\mathfrak{N}$ closed

$A \subset \mathfrak{H} \times \mathfrak{H}$ is range space relation (RSR) if

A is a range subspace of $\mathfrak{H} \times \mathfrak{H}$

If A, B are RSR

then $A + B$, AB are RSR and

ker A, mul A, ran A, dom A range subspaces

- if dom A closed then ker A closed

- if ran A closed then mul A closed

- if ker A and ran A closed then A closed

- if mul A and dom A closed then A closed

The degree of a relation

$\Delta(A) =$

$\{n \in \mathbb{N} : \text{ran}\, A^n \cap \ker A = \text{ran}\, A^m \cap \ker A, \ \forall\, m \geq n\}$

$\delta(A) = \min \Delta(A) \quad \text{if} \quad \Delta(A) \neq \emptyset$

$\delta(A) = \infty \quad \text{if} \quad \Delta(A) = \emptyset$

Equivalent are

- $d \in \Delta(A)$
- $\ker A^m \subseteq \ker A^d + \text{ran}\, A^n, \quad m, n \in \mathbb{N}$
- $\text{ran}\, A^d \cap \ker A^n \subseteq \text{ran}\, A^m \cap \ker A^n, \quad m, n \in \mathbb{N}$

Quasi-Fredholm relations

RSR A in $\mathfrak{H}$ is quasi-Fredholm of degree d if

(Q1) $d = \delta(A) < \infty$

(Q2) $\operatorname{ran} A^d \cap \ker A$ closed in $\mathfrak{H}$;

(Q3) $\operatorname{ran} A + \ker A^d$ closed in $\mathfrak{H}$.

A is quasi-Fredholm of degree 0 iff

$\ker A \subset \operatorname{ran} A^m$, $m \in \mathbb{N}$

and $\ker A$ is closed in $\mathfrak{H}$

and $\operatorname{ran} A$ is closed in $\mathfrak{H}$

If A is quasi-Fredholm of degree 1 then

$\ker A$, $\operatorname{ran} A$ are closed by Neubauer's Lemma

Theorem

Let A be quasi-Fredholm of degree d.

Then $\exists$ closed subspaces $\mathfrak{M}, \mathfrak{N} \subset \mathfrak{H}$

(K1) $\mathfrak{H} = \mathfrak{M} + \mathfrak{N}$, $\mathfrak{M} \cap \mathfrak{N} = \{0\}$

(K2) $\operatorname{ran} A^d \subseteq \mathfrak{M}$

(K3) $\mathfrak{N} \subseteq \ker A^d$, if $d \geq 1$, $\mathfrak{N} \not\subset \ker A^{d-1}$

(K4) $A = A_{\mathfrak{M}\times\mathfrak{M}} \widehat{+} A_{\mathfrak{N}\times\mathfrak{N}}$, direct sum

(K5) $A_{\mathfrak{M}\times\mathfrak{M}}$ quasi-Fredholm of degree 0

(K6) $(A_{\mathfrak{N}\times\mathfrak{N}})^d = \mathfrak{N} \times \{0\}$

Construction of $\mathfrak{M}$ and $\mathfrak{N}$

$\mathfrak{M}_0 = \mathfrak{H}$

$\mathfrak{M}_{j+1} = (\text{ran}\, A + \text{ker}\ A^d)^\perp + \left\{ v : \{u, v\} \in A,\ u \in \mathfrak{M}_j \right\}$

$\mathfrak{M}_{j+1} \subset \mathfrak{M}_j$

$\mathfrak{N}_0 = \{0\}$

$\mathfrak{N}_{j+1} = \left\{ u \in (\text{ran}\, A^d \cap \text{ker}\ A)^\perp : \{u, v\} \in A,\ v \in \mathfrak{N}_j \right\}$

$\mathfrak{N}_j \subset \mathfrak{N}_{j+1}$

$\mathfrak{M}_d = \mathfrak{M}_j$, $\mathfrak{N}_d = \mathfrak{N}_j$ for $j \geq d$

$\mathfrak{M} = \mathfrak{M}_d$, $\mathfrak{N} = \mathfrak{N}_d$

Theorem

Let A be quasi-Fredholm of degree d. Then

- ran A^n + ker A^m closed for $n + m \geq d$

- mul A^n closed for $n \in \mathbb{N}$

Decompositions

RSR A in $\mathfrak{H}$ is Kato decomposable

of degree d if $\exists$ closed subspaces $\mathfrak{M}, \mathfrak{N} \subset \mathfrak{H}$

such that (K1–6) are satisfied.

RSR A in $\mathfrak{H}$ is normally decomposable

of degree d if $\exists$ RSR D, operator T with

(N1) $A = D + T$, $TD = \mathrm{dom}\, A \times \{0\}$
$DT = \mathfrak{H} \times \mathrm{mul}\, A$;

(N2) D quasi-Fredholm of degree ≤ 1

(N3) $T^d = 0$

Theorem

Equivalent are

- A is Kato decomposable of degree d

- A is normally decomposable of degree d

- A is quasi-Fredholm of degree d

Remark

$$D = A_{\mathfrak{M}\times\mathfrak{M}} P_{\mathfrak{M}}, \quad T = A_{\mathfrak{N}\times\mathfrak{N}} P_{\mathfrak{N}}$$

If A is quasi-Fredholm then A is closed

Theorem

Let A be a quasi-Fredholm of degree d. Then

- A^* is quasi-Fredholm relation of degree d

- $\text{ran}\, A^n + \text{ker}\ A^m = (\text{ker}\ A^{*n} \cap \text{ran}\, A^{*m})^\perp$
  $n + m \geq d$

- $\text{ran}\, A^{*n} + \text{ker}\ A^{*m} = (\text{ker}\ A^n \cap \text{ran}\, A^m)^\perp$
  $n + m \geq d$

- $(\text{ker}\ A^n)^\perp = \text{ran}\, A^{*n}$, $(\text{ker}\ A^{*n})^\perp = \text{ran}\, A^n$
  $n \geq d$

Semi-Fredholm relations

Closed relation A in $\mathfrak{H}$ is semi-Fredholm if

(S1) $\operatorname{ran} A$ closed;

(S2) $\dim \ker A < \infty$ or $\dim (\mathfrak{H}/\operatorname{ran} A) < \infty$

Semi-Fredholm implies quasi-Fredholm

If $\dim \ker A < \infty$

$\dim (\ker A \cap \operatorname{ran} A^n)$ nonincreasing

$\exists\ d$ with $\ker A \cap \operatorname{ran} A^d = \ker A \cap \operatorname{ran} A^{d+k}$

$\dim \ker A^d < \infty \Longrightarrow \operatorname{ran} A + \ker A^d$ closed

If $\dim (\mathfrak{H}/\operatorname{ran} A) < \infty$

$\dim (\ker A/(\ker A \cap \operatorname{ran} A^m))$

$= \dim (\ker A^m + \operatorname{ran} A)/\operatorname{ran} A$

$\leq \dim (\mathfrak{H}/\operatorname{ran} A) < \infty$