Domination in Krein spaces

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I. Motivation.

$[A, B] := AB - BA$ (on the maximal domain)

THEOREM 1 [Cichoń,Stochel,Szafraniec 2004]
$m \in \mathbb{N}$;
S-selfadjoint, A-symmetric in a Hilbert space,
$\mathcal{D}(S^m) \subseteq \mathcal{D}(A)$, and

$$(1) \qquad \sup_{n\in\mathbb{N}} \left\| \, [n^m(S - n\,\mathrm{i})^{-m}, A] \, \right\| < +\infty.$$

Then A is essentially selfadjoint on any core of S^m.

Note: 1. $\mathcal{D}(S^m) \subseteq \mathcal{D}(A)$ implies that

$$\|Af\| \le c\,\|f\| + d\,\|S^m f\|\,, \quad f \in \mathcal{D}(S^m)\,.$$

We do *not* assume that $d < 1$.

2. If A is e.g. a bounded perturbation of S, then (1) holds.

3. Instead of $n^m(S - n\,\mathrm{i})^{-m}$ we could take $|z_n|^m(S - z_n)^{-m}$, where $(z_n)_{n=1}^{\infty}$ tends nontangentially to infinity.

II. Main thoeorem.

DEFINITION. $\mathcal{K}$-Krein space, A-symmetric in $\mathcal{K}$;
$(S_n)_{n=0}^{\infty} \subseteq \mathbf{B}(\mathcal{K})$ *dominates* A if

- $\forall n \in \mathbb{N} \quad [S_n, A]$ is densely defined,
- $\overline{S_n A}, \overline{A S_n} \in \mathbf{B}(\mathcal{K})$;
- $\forall n \in \mathbb{N} \quad A S_n^+$ is densely defined;
- $\textsc{wot}\lim_{n\to\infty} S_n = I_{\mathcal{K}}$.

THEOREM 2. A-symmetric operator in a Krein space $\mathcal{K}$,
$(S_n)_{n=0}^{\infty} \subseteq \mathbf{B}(\mathcal{K})$ dominates A and

$$\sup_{n\in\mathbb{N}} \left\| \overline{[S_n, A]} \right\| < +\infty. \tag{2}$$

Then A is essentially selfadjoint in $\mathcal{K}$.

III. Instances.

1. S-selfadjoint, A-symmetric in a Hilbert space $\mathcal{H}$, $\mathcal{D}(S^m) \subseteq \mathcal{D}(A)$ for some $m \in \mathbb{N}$. Then

$$S_n := n^m (S - n\,\mathrm{i})^{-m}, \quad n = 1, 2 \ldots$$

dominates A. (2)=(1). This together with Thm2. gives us a different proof of Thm1.

2. S-selfadjoint, A-symmetric in a Hilbert space $\mathcal{H}$,
$\mathcal{D}(S^m) \subseteq \mathcal{D}(A)$ for some $m \in \mathbb{N}$,
K is S-compact,

$$\mathcal{R}(K) \cup \mathcal{R}(K^*) \subseteq \mathcal{D}\left(S^{m-1}\right).$$

Then the sequence

$S_n := n^m (S + K - n\,\mathrm{i})^{-m}, \quad n = k+1, k+2 \ldots$

dominates A.

3. In particular, if $\mathcal{H} = \mathcal{H}_0 \times \mathcal{H}_1$,

$$S := \begin{pmatrix} S_0 & 0 \\ 0 & S_1 \end{pmatrix}, \quad K := \begin{pmatrix} 0 & -K_0^* \\ K_0 & 0 \end{pmatrix}$$

where $S_i = S_i^*$ $(i = 0, 1)$,
K_0 is S_0-compact, K_0^* is S_1-compact,
$\mathcal{R}(K_0) \subseteq \mathcal{D}\left(S_1^{m-1}\right)$, $\mathcal{R}(K_0^*) \subseteq \mathcal{D}\left(S_0^{m-1}\right)$.

Since every selfadjoint operator in a Pontriagin space is of the form $S + K$, Thm1 holds in a Pontriagin space.

4. S-selfadjoint, A-symmetric in a Krein space $\mathcal{K}$, S similar to a selfadjoint operator in a Hilbert space, $\mathcal{D}\left(S^m\right) \subseteq \mathcal{D}\left(A\right)$ for some $m \in \mathbb{N}$. Then

$$S_n := n^m(S - n\,\mathrm{i})^{-m}, \quad n = 1, 2 \ldots$$

dominates A.

5. $(S_n)_{n=1}^{\infty}$ is a sequence of projections tending in WOT to $I_{\mathcal{K}}$, e.g. $S_n := E(\{|z| < n\})$, where E is a spectral function of a normal operator in a Hilbert space ([Nussbaum, 1969]).

IV. More on commutativity.

Theorem 3.S-selfadjoint, A-symmetric in a Krein space,
$\mathcal{D}\left(S^m\right) \subseteq \mathcal{D}\left(A\right)$, and
$\mathcal{D}\left(S\right) \subseteq \mathcal{D}\left(\overline{[S, A]}\right)$ then

$$\sup_{n \in \mathbb{N}} \left\| \, [n^m(S - n\,\mathrm{i})^{-m}, A] \, \right\| < \infty. \tag{3}$$

V. Example

$\mathcal{H}$-Hilbert space,
$J : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$, $J(f, g) := (g, f)$;
N - normal in $\mathcal{H}$; $\rho(N) = \emptyset$.

$$S := \begin{pmatrix} \mathrm{Re}N & 0 \\ 0 & \mathrm{Re}N \end{pmatrix}, \quad A := \begin{pmatrix} N & 0 \\ 0 & N^* \end{pmatrix}.$$

A is symmetric in the Krein space given by J.
The sequence $S_n := n(S - n\,\mathrm{i})^{-1}$ and operator A satisfy all the assumptions of Thm2, A is selfadjoint, and $\rho(A) = \emptyset$.