Constructions and existence of tight fusion frames

Peter G. Casazza[a], Matthew Fickus[b], Dustin G. Mixon[c], Yang Wang[d] and Zhengfang Zhou[d]

[a]Department of Mathematics, University of Missouri, Columbia, Missouri 65211, USA
[b]Department of Mathematics and Statistics, Air Force Institute of Technology
Wright-Patterson Air Force Base, Ohio 45433, USA
[c]Program in Applied and Computational Mathematics, Princeton University
Princeton, New Jersey 08544, USA
[d]Department of Mathematics, Michigan State University, East Lansing, Michigan 48824, USA

ABSTRACT

Fusion frames are an emerging topic of frame theory, with applications to communications and distributed processing. However, until recently, little was known about the existence of tight fusion frames, much less how to construct them. We discuss a new method for constructing tight fusion frames which is akin to playing Tetris with the spectrum of the frame operator. When combined with some easily obtained necessary conditions, these Spectral Tetris constructions provide a near complete characterization of the existence of tight fusion frames.

Keywords: tight, fusion, frames

1. INTRODUCTION

A *tight fusion frame* (TFF) is a sequence of orthogonal projections that sum to a scalar multiple of the identity. Following their introduction,[5,7] such frames have been shown to be robust against additive noise and erasures.[2,6,9] As such, tight fusion frames are well-suited for emerging real-world applications in communications and distributed sensing.[8,11,12] In this paper, we discuss a near-complete characterization of the existence of tight fusion frames in the special case where all of the projections are of equal rank; it may be shown[2] that such TFFs provide optimal blind-reconstruction in the case where a single projection's worth of data is erased.

This paper contains but a brief sketch of the main ideas and results that lie behind our near-complete characterization, along with some illustrative examples; for the complete characterization, as well as the proofs of the results presented here, we refer the interested reader to the accompanying journal article.[4] In the next section, we discuss how TFFs can be regarded as special cases of unit norm tight frames, and consider a method for constructing new TFFs as the orthogonal complements of existing ones. In Section 3, we consider a new fundamental technique for constructing unit norm tight frames. This method resembles the popular game Tetris[TM], as it involves building a flat spectrum with blocks of fixed area. We then modulate this *Spectral Tetris* construction to produce *Gabor fusion frames*.

2. PRELIMINARIES AND NOTATION

The *synthesis operator* of a finite sequence of vectors $\{f_m\}_{m=1}^M$ in $\mathbb{C}^N$ is $F:\mathbb{C}^M\rightarrow\mathbb{C}^N$,

$$Fg=\sum_{m=1}^{M} g(m)f_m.$$

That is, F is an $N\times M$ matrix whose mth column is f_m. The corresponding *analysis operator* is $F^*:\mathbb{C}^N\rightarrow\mathbb{C}^M$, $(F^*f)(m)=\langle f,f_m\rangle$, while the corresponding *frame operator* is $FF^*:\mathbb{C}^N\rightarrow\mathbb{C}^N$,

$$FF^*f=\sum_{m=1}^{M}\langle f,f_m\rangle f_m.$$

Send correspondence to M.F.: E-mail: Matthew.Fickus@afit.edu

Generally speaking, *frame theory* is the study of how $\{f_m\}_{m=1}^M$ should be chosen so as to ensure that FF^* is well-conditioned. In particular, $\{f_m\}_{m=1}^M$ is a *tight frame* if there exists $A > 0$ such that $FF^* = A\mathrm{I}$, namely that:

$$Af = \sum_{m=1}^{M} \langle f, f_m \rangle f_m \tag{1}$$

for all $f \in \mathbb{C}^N$. Since, letting $\{e_n\}_{n=1}^N$ be the standard basis for $\mathbb{C}^N$, one may note:

$$\langle FF^* e_n, e_{n'} \rangle = \langle F^* e_n, F^* e_{n'} \rangle = \sum_{m=1}^{M} (F^* e_n)(m)\overline{(F^* e_{n'})(m)} = \sum_{m=1}^{M} \langle f_m, e_n \rangle \overline{\langle f_m, e_{n'} \rangle} = \sum_{m=1}^{M} f_m(n)\overline{f_m(n')},$$

we equivalently have that $\{f_m\}_{m=1}^M$ is A-tight if:

$$\sum_{m=1}^{M} f_m(n)\overline{f_m(n')} = \begin{cases} A, & n = n', \\ 0, & n \neq n'. \end{cases}$$

A *unit norm tight frame* (UNTF) is a tight frame $\{f_m\}_{m=1}^M$ which further satisfies $\|f_m\| = 1$ for all $m = 1, \dots, M$. UNTFs provide Parseval-like decompositions in terms of nonorthogonal vectors of unit norm.

Fusion frame theory generalizes these concepts. In particular, when each f_m is of unit norm, the summands of (1), namely, the operators $f \mapsto \langle f, f_m \rangle f_m$, are rank-one orthogonal projections. Fusion frame theory is the study of sums of projections whose ranks are not necessarily restricted to be one. In particular, we say that a sequence $\{P_k\}_{k=1}^K$ of $N \times N$ orthogonal projection matrices of rank L is a (K, L, N)-TFF if there exists $A > 0$ such that:

$$A\mathrm{I} = \sum_{k=1}^{K} P_k. \tag{2}$$

From this perspective, fusion frame theory may be regarded as a generalization of classical frame theory. Alternatively, fusion frames may be regarded as special cases of traditional frames; to elaborate, note that for any fixed $k = 1, \dots, K$, we classically know that P_k is the frame operator for any orthonormal basis $\{f_{k,l}\}_{l=1}^L$ of its range, that is,

$$P_k f = \sum_{l=1}^{L} \langle f, f_{k,l} \rangle f_{k,l}$$

for all $f \in \mathbb{C}^N$. Summing these equations over $k = 1, \dots, K$ yields:

$$\sum_{k=1}^{K} P_k f = \sum_{k=1}^{K} \sum_{l=1}^{L} \langle f, f_{k,l} \rangle f_{k,l}. \tag{3}$$

In light of (1) and (2), (3) implies that every TFF arises from a traditional tight frame in which the frame vectors themselves are required to satisfy additional orthogonality requirements. To be precise:

Definition 1. A sequence $\{f_{k,l}\}_{k=1,\,l=1}^{K,\;\,L} \subset \mathbb{C}^N$ *generates* a (K, L, N)-TFF if:

i. $\{f_{k,l}\}_{l=1}^L$ is orthonormal for every $k = 1, \dots, K$.

ii. $\{f_{k,l}\}_{k=1,\,l=1}^{K,\;\,L}$ is a tight frame for $\mathbb{C}^N$, that is, there exists $A > 0$ such that for any $f \in \mathbb{C}^N$,

$$Af = \sum_{k=1}^{K} \sum_{l=1}^{L} \langle f, f_{k,l} \rangle f_{k,l}. \tag{4}$$

Equivalently, the rows of the synthesis operator are mutually orthogonal with equal norm:

$$\sum_{k=1}^{K} \sum_{l=1}^{L} f_{k,l}(n)\overline{f_{k,l}(n')} = \begin{cases} A, & n = n', \\ 0, & n \neq n'. \end{cases} \tag{5}$$

From this perspective, we see that (K, L, N)-TFFs are actually special cases of UNTFs of KL elements for $\mathbb{C}^N$, and as such, the tight frame constant in (4) is necessarily[1] $A = \frac{KL}{N}$.

We exploit this UNTF-based representation of TFFs to characterize the existence of (K, L, N)-TFFs. In particular, as any UNTF is necessarily a spanning set, the existence of (K, L, N)-TFFs necessarily requires that $KL \geq N$. Moreover, this necessary result is also sufficient in the special case where L divides N: one may construct (K, L, N)-TFFs by taking direct sums of L copies of a UNTF of K elements for $\mathbb{C}^{\frac{N}{L}}$. In short:

Theorem 2. *If L divides N, then (K, L, N)-TFFs exist if and only if $K \geq \frac{N}{L}$.*

As such, the remainder of this paper is devoted to the case where L does not divide N. In this case, we can show that a stronger condition, namely that $K \geq \lceil \frac{N}{L} \rceil + 1$, is, in fact, necessary. Indeed, if $KL \geq N$ and L does not divide N, then $KL > N$. In this case, we can take a generating sequence for a given (K, L, N)-TFF, extend its $N \times KL$ synthesis matrix to a $KL \times KL$ unitary matrix, and then consider the $(N - KL) \times KL$ extension. We claim these new vectors also generate a TFF, termed the *Naimark complement* of the original, as the construction makes use of Naimark's argument that every 1-tight frame is the projection of an orthonormal basis. Here, the number and dimension of the new TFF's subspaces are equal to those of the original, but the dimension of the underlying space changes:

Theorem 3. *If $N < KL$ and $\{f_{k,l}\}_{k=1,\ l=1}^{K\ \ \ L}$ generates a (K, L, N)-TFF, then any $\{g_{k,l}\}_{k=1,\ l=1}^{K\ \ \ L} \subset \mathbb{C}^{KL-N}$ for which:*

$$\{\tfrac{\sqrt{N}}{\sqrt{KL}} f_{k,l} \oplus \tfrac{\sqrt{KL-N}}{\sqrt{KL}} g_{k,l}\}_{k=1,\ l=1}^{K\ \ \ L} \text{ is an orthonormal basis for } \mathbb{C}^{KL}$$

generates a $(K, L, KL - N)$-TFF.

In particular, if (K, L, N)-TFFs exist but L does not divide N, then there exists $\{g_{k,l}\}_{k=1,\ l=1}^{K\ \ \ L} \subset \mathbb{C}^{KL-N}$ such that for any $k = 1, \ldots, K$, the collection $\{g_{k,l}\}_{l=1}^{L} \subset \mathbb{C}^{KL-N}$ is orthonormal. Thus, if (K, L, N)-TFFs exist, then we necessarily have that $L \leq KL - N$, that is, $K \geq \frac{N}{L} + 1$. Since K is an integer, this further implies:

Corollary 4. *If (K, L, N)-TFFs exist and L does not divide N, then $K \geq \lceil \frac{N}{L} \rceil + 1$.*

Moreover, one may show that the necessary condition of Corollary 4 is not sufficient. In fact, a simple argument[4] shows that $(4, 4, 11)$-TFFs do not exist, despite the fact that $4 \geq \lceil \frac{11}{4} \rceil + 1$. Fortunately, we can show that such counterexamples are the exception, rather than the rule. That is, a slight strengthening of this necessary condition is indeed sufficient:

Theorem 5. *If $2L < N$, L does not divide N, and $K \geq \lceil \frac{N}{L} \rceil + 2$, then (K, L, N)-TFFs exist.*

Moreover, the proof of this result, discussed in the following section, is entirely constructive.

3. CONSTRUCTING TIGHT FUSION FRAMES WITH SPECTRAL TETRIS

In this section, we provide a general method for constructing (K, L, N)-TFFs when $K \geq \lceil \frac{N}{L} \rceil + 2$. The key idea is to revisit the simpler problem of constructing UNTFs, that is, sequences $\{f_m\}_{m=1}^{M}$ of unit vectors in $\mathbb{C}^N$ that satisfy (1). In brief, we want to construct $N \times M$ synthesis matrices F which have:

i. columns of unit norm,

ii. orthogonal rows, meaning the frame operator FF^* is diagonal,

iii. rows of constant norm, meaning FF^* is a constant multiple of the identity matrix.

Despite a decade of study, very few general constructions of UNTFs are known. Moreover, these known methods unfortunately manipulate all frame elements simultaneously. In this section, we discuss how constructing certain examples of UNTFs need not be so difficult. In particular, we provide a new, iterative method for constructing UNTFs, building them one or two vectors at a time. The key idea is to iteratively build a matrix F which, at each iteration, exactly satisfies (i) and (ii), and gets closer to satisfying (iii). We call this method *Spectral Tetris*, as it involves building a flat spectrum out of blocks of fixed area. Here, an illustrative example is helpful:

Example 6. In the previous section, we noted that $(4,4,11)$-TFFs did not exist, despite the fact that these K, L and N satisfy the necessary condition for existence given in Corollary 4. At the same time, we claim in Theorem 5 that a slightly stronger requirement, $K \geq \lceil \frac{N}{L} \rceil + 2$, is indeed sufficient for existence, provided L does not divide N and $2L < N$. In particular, Theorem 5 asserts that $(5,4,11)$-TFFs exist. In this paper, we will show how to explicitly construct such a TFF, so as to illustrate the simple ideas behind the proof of Theorem 5. The construction is performed over two stages. The first stage, given in the present example, is to play Spectral Tetris, yielding a sparse UNTF of 11 elements for $\mathbb{C}^4$. In the second stage, this UNTF is then modulated to produce a $(5,4,11)$-TFF, as described in Example 11.

Our immediate goal is to create a 4×11 matrix F such that $FF^* = \frac{11}{4}\mathrm{I}$. As such, we begin with an arbitrary 4×11 matrix, and, without loss of generality, let the first frame element be the first standard basis element e_1, namely:

$$F = \left[\begin{array}{ccccccccccc} 1 & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ 0 & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ 0 & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ 0 & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \end{array}\right], \tag{6}$$

and so the corresponding frame operator is of the form:

$$FF^* = \left[\begin{array}{cccc} 1+? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right].$$

If the remaining unknown entries are chosen so that F has orthogonal rows, then FF^* will be a diagonal matrix. Currently, the diagonal entries of FF^* are mostly unknown, having the form $\{1+?, ?, ?, ?\}$. Also note that if the remainder of the first row of F is set to zero, then the first diagonal entry of FF^* would be $1 < \frac{11}{4}$. Thus, we need to add more weight to this row. In particular, making the second column of F another copy of e_1 yields:

$$F = \left[\begin{array}{ccccccccccc} 1 & 1 & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? & ? & ? & ? & ? & ? \end{array}\right],$$

with frame operator:

$$FF^* = \left[\begin{array}{cccc} 2+? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right].$$

Now, if the remainder of the first row of F is set to zero, then the first diagonal entry of FF^* falls short: $2 < \frac{11}{4}$. Thus, as before, we need to add more weight to this row. However, making the third column of F another copy of e_1 would add too much weight, as $3 > \frac{11}{4}$. Therefore, we need a way to put $\frac{11}{4} - 2 = \frac{3}{4}$ more weight in the first row without compromising the orthogonality of the rows of F nor the normality of its columns. The key idea is to realize that for any $0 \leq x \leq 2$, there exists a 2×2 matrix $T(x)$ with orthogonal rows and unit-length columns such that $T(x)T^*(x)$ is a diagonal matrix with diagonal entries $\{x, 2-x\}$. Specifically, we have:

$$T(x) := \tfrac{1}{\sqrt{2}} \left[\begin{array}{cc} \sqrt{x} & \sqrt{x} \\ \sqrt{2-x} & -\sqrt{2-x} \end{array}\right], \quad T(x)T^*(x) = \left[\begin{array}{cc} x & 0 \\ 0 & 2-x \end{array}\right].$$

We define the third and fourth columns of F according to such a matrix $T(x)$, where $x = \frac{11}{4} - 2 = \frac{3}{4}$:

$$F = \left[\begin{array}{ccccccccccc} 1 & 1 & \frac{\sqrt{3}}{\sqrt{8}} & \frac{\sqrt{3}}{\sqrt{8}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{5}}{\sqrt{8}} & -\frac{\sqrt{5}}{\sqrt{8}} & ? & ? & ? & ? & ? & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? & ? & ? & ? & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? & ? & ? & ? & ? & ? \end{array}\right], \tag{7}$$

yielding a frame operator of the form:

$$FF^* = \begin{bmatrix} \frac{11}{4} & 0 & 0 & 0 \\ 0 & \frac{5}{4}+? & ? & ? \\ 0 & ? & ? & ? \\ 0 & ? & ? & ? \end{bmatrix}.$$

The first row now has sufficient weight, and so its remaining entries are set to zero. The second entry is currently falling short by $\frac{11}{4} - \frac{5}{4} = \frac{6}{4} > 1$, and as such, we make the fifth column e_2:

$$F = \begin{bmatrix} 1 & 1 & \frac{\sqrt{3}}{\sqrt{8}} & \frac{\sqrt{3}}{\sqrt{8}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{5}}{\sqrt{8}} & -\frac{\sqrt{5}}{\sqrt{8}} & 1 & ? & ? & ? & ? & ? & ? \\ 0 & 0 & 0 & 0 & 0 & ? & ? & ? & ? & ? & ? \\ 0 & 0 & 0 & 0 & 0 & ? & ? & ? & ? & ? & ? \end{bmatrix},$$

with frame operator:

$$FF^* = \begin{bmatrix} \frac{11}{4} & 0 & 0 & 0 \\ 0 & \frac{9}{4}+? & ? & ? \\ 0 & ? & ? & ? \\ 0 & ? & ? & ? \end{bmatrix}.$$

As the second diagonal entry of FF^* is falling short by $\frac{2}{4} < 1$, we cannot let the sixth column be e_2, but rather let the sixth and seventh columns arise from $T(\frac{2}{4})$:

$$F = \begin{bmatrix} 1 & 1 & \frac{\sqrt{3}}{\sqrt{8}} & \frac{\sqrt{3}}{\sqrt{8}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{5}}{\sqrt{8}} & -\frac{\sqrt{5}}{\sqrt{8}} & 1 & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}} & ? & ? & ? & ? \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & ? & ? & ? & ? \end{bmatrix}, \tag{8}$$

yielding a frame operator of the form:

$$FF^* = \begin{bmatrix} \frac{11}{4} & 0 & 0 & 0 \\ 0 & \frac{11}{4} & 0 & 0 \\ 0 & 0 & \frac{6}{4}+? & ? \\ 0 & 0 & ? & ? \end{bmatrix}.$$

Continuing in this manner, we take the eighth column of F to be e_3:

$$F = \begin{bmatrix} 1 & 1 & \frac{\sqrt{3}}{\sqrt{8}} & \frac{\sqrt{3}}{\sqrt{8}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{5}}{\sqrt{8}} & -\frac{\sqrt{5}}{\sqrt{8}} & 1 & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}} & 1 & ? & ? & ? \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ? & ? & ? \end{bmatrix},$$

yielding the frame operator:

$$FF^* = \begin{bmatrix} \frac{11}{4} & 0 & 0 & 0 \\ 0 & \frac{11}{4} & 0 & 0 \\ 0 & 0 & \frac{10}{4}+? & ? \\ 0 & 0 & ? & ? \end{bmatrix};$$

let the ninth and tenth columns arise from $T(\frac{1}{4})$:

$$F = \begin{bmatrix} 1 & 1 & \frac{\sqrt{3}}{\sqrt{8}} & \frac{\sqrt{3}}{\sqrt{8}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{5}}{\sqrt{8}} & -\frac{\sqrt{5}}{\sqrt{8}} & 1 & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}} & 1 & \frac{\sqrt{1}}{\sqrt{8}} & \frac{\sqrt{1}}{\sqrt{8}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{\sqrt{8}} & -\frac{\sqrt{7}}{\sqrt{8}} & ? \end{bmatrix},$$

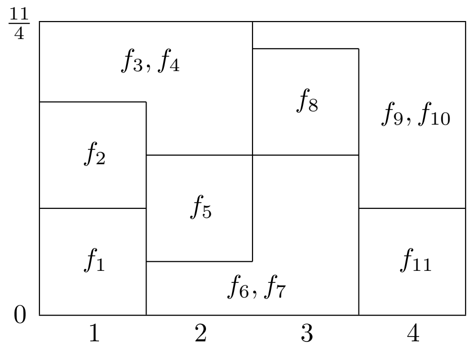

Figure 1. The Spectral Tetris construction of a UNTF of 11 elements for $\mathbb{C}^4$, as detailed in Example 6. Each of the four columns corresponds to a diagonal entry of the frame operator FF^*, and each block represents the contribution made to these entries by the corresponding frame elements. For example, the single frame element $\{f_2\}$ contributes $\{1, 0, 0, 0\}$ to the diagonal, while the pair $\{f_6, f_7\}$ contributes $\{0, \frac{2}{4}, \frac{6}{4}, 0\}$. The area of the blocks is determined by the number of frame elements that generate them: blocks that arise from a single element have unit area, while blocks that arises from two elements have an area of 2. In order for $\{f_m\}_{m=1}^{11}$ to be a UNTF for $\mathbb{C}^4$, these blocks needed to stack to a uniform height of $\frac{11}{4}$. By building a rectangle from blocks of given areas, we are essentially playing Tetris with the spectrum of FF^*.

yielding:

$$FF^* = \begin{bmatrix} \frac{11}{4} & 0 & 0 & 0 \\ 0 & \frac{11}{4} & 0 & 0 \\ 0 & 0 & \frac{11}{4} & 0 \\ 0 & 0 & 0 & \frac{7}{4}+? \end{bmatrix}.$$

We conclude by letting the final column be e_4,

$$F = \begin{bmatrix} 1 & 1 & \frac{\sqrt{3}}{\sqrt{8}} & \frac{\sqrt{3}}{\sqrt{8}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{5}}{\sqrt{8}} & -\frac{\sqrt{5}}{\sqrt{8}} & 1 & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}} & 1 & \frac{\sqrt{7}}{\sqrt{8}} & \frac{\sqrt{7}}{\sqrt{8}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{\sqrt{8}} & -\frac{\sqrt{7}}{\sqrt{8}} & 1 \end{bmatrix}, \tag{9}$$

which indeed corresponds to a UNTF:

$$FF^* \begin{bmatrix} \frac{11}{4} & 0 & 0 & 0 \\ 0 & \frac{11}{4} & 0 & 0 \\ 0 & 0 & \frac{11}{4} & 0 \\ 0 & 0 & 0 & \frac{11}{4} \end{bmatrix}.$$

In this construction, column vectors are either introduced one at a time, such as $\{f_1\}$, $\{f_2\}$, $\{f_5\}$, $\{f_8\}$ or $\{f_{11}\}$, or in pairs, such as $\{f_3, f_4\}$, $\{f_6, f_7\}$ or $\{f_9, f_{10}\}$. Each singleton contributes a value of 1 to a particular diagonal entry of FF^*, while each pair spreads two units of weight over two entries. Overall, we have formed a flat spectrum, $\{\frac{11}{4}, \frac{11}{4}, \frac{11}{4}, \frac{11}{4}\}$, from blocks of area 1 or 2. This construction is reminiscent of the game Tetris, as illustrated in Figure 1.

We conclude this example by pointing out a crucial consequence of this Spectral Tetris construction: the frame vectors are extremely sparse, with many pairs of vectors having mutually disjoint support. In particular, we have that f_m and $f_{m'}$ are orthogonal whenever $m - m' \geq 5$. More generally, we can show that whenever Spectral Tetris is played to form a UNTF, the resulting frame elements satisfy $\langle f_m, f_{m'} \rangle = 0$ whenever $m' - m \geq \lfloor \frac{M}{N} \rfloor + 3$. These orthogonality relations will play a critical role in the next subsection, where Spectral Tetris UNTFs will be modulated to form Gabor TFFs.

In order to formalize the Spectral Tetris argument used in the previous example, we introduce the following notion:

Definition 7. We say that a sequence $\{f_m\}_{m=1}^M$ is an (m_0,n_0)-*proto unit norm tight frame* (PUNTF) for $\mathbb{C}^N$ if:

i. $\displaystyle\sum_{n=1}^N |f_m(n)|^2 = \begin{cases} 1, & m \le m_0,\\ 0, & m > m_0,\end{cases}$

ii. $\displaystyle\sum_{m=1}^M f_m(n)\overline{f_m(n')} = 0$ for all $n, n' = 1,\dots,N, n \neq n'$,

iii. $\displaystyle\sum_{m=1}^M |f_m(n)|^2 = \begin{cases} \frac{M}{N}, & n < n_0,\\ 0, & n > n_0,\end{cases}$

iv. $\displaystyle 1 \le \sum_{m=1}^M |f_m(n_0)|^2 \le \frac{M}{N}.$

That is, $\{f_m\}_{m=1}^M$ is an (m_0,n_0)-PUNTF for $\mathbb{C}^N$ precisely when its $N \times M$ synthesis matrix F vanishes off of its upper-left $n_0 \times m_0$ submatrix, its nonzero columns have unit norm, and its frame operator FF^* is diagonal, with the first $n_0 - 1$ diagonal entries being $\frac{M}{N}$, the n_0th entry lying in $[1, \frac{M}{N}]$, and the remaining entries being zero. In particular, setting "?" entries to zero in (6), (7), (8) and (9) results in $(2,1)$-, $(4,2)$-, $(7,3)$- and $(11,4)$-PUNTFs, respectively. As seen in Example 6, the goal of Spectral Tetris is to iteratively create larger PUNTFs from existing ones, continuing until $(m_0,n_0) = (M,N)$, at which point the PUNTF is a UNTF. We now give the precise rules for enlarging a given PUNTF:

Theorem 8. *Let $2N \le M$, let $\{f_m\}_{m=1}^M$ be an (m_0,n_0)-PUNTF for $\mathbb{C}^N$, and let $\lambda := \displaystyle\sum_{m=1}^M |f_m(n_0)|^2$.*

i. If $\lambda \le \frac{M}{N} - 1$, then $m_0 < M$ and $\{g_m\}_{m=1}^M$,

$$g_m := \begin{cases} f_m, & m \le m_0,\\ e_{n_0}, & m = m_0+1,\\ 0, & m > m_0+1,\end{cases}$$

is an (m_0+1, n_0)-PUNTF for $\mathbb{C}^N$.

ii. If $\frac{M}{N} - 1 < \lambda < \frac{M}{N}$, then $m_0 < M-2$, $n_0 < N$ and $\{g_m\}_{m=1}^M$,

$$g_m := \begin{cases} f_m, & m \le m_0,\\ \sqrt{\tfrac{1}{2}(\tfrac{M}{N}-\lambda)}e_{n_0} + \sqrt{1-\tfrac{1}{2}(\tfrac{M}{N}-\lambda)}e_{n_0+1}, & m = m_0+1,\\ \sqrt{\tfrac{1}{2}(\tfrac{M}{N}-\lambda)}e_{n_0} - \sqrt{1-\tfrac{1}{2}(\tfrac{M}{N}-\lambda)}e_{n_0+1}, & m = m_0+2,\\ 0, & m > m_0+2,\end{cases}$$

is an (m_0+2, n_0+1)-PUNTF for $\mathbb{C}^N$.

iii. If $\lambda = \frac{M}{N}$ and $n_0 < N$, then $m_0 < M$ and $\{g_m\}_{m=1}^M$,

$$g_m := \begin{cases} f_m, & m \le m_0,\\ e_{n_0+1}, & m = m_0+1,\\ 0, & m > m_0+1,\end{cases}$$

is an (m_0+1, n_0+1)-PUNTF for $\mathbb{C}^N$.

iv. If $\lambda = \frac{M}{N}$ and $n_0 = N$, then $\{f_m\}_{m=1}^M$ is a UNTF for $\mathbb{C}^N$.

The assumption $2N \le M$ is crucial to the proof of Theorem 8; in the case where λ is slightly smaller than $\frac{M}{N}$, the (n_0+1)th diagonal entry of FF^* must accept nearly two spectral units of weight, which is only possible when the desired Spectral Tetris height $\frac{M}{N}$ is at least 2. At the same time, we note that playing Spectral Tetris can also result in matrices of lesser redundancy, provided larger blocks are used. Indeed, UNTFs of redundancy $\frac{M}{N} \ge \frac{3}{2}$ can be constructed using 3×3 Spectral Tetris submatrices, as we now have two diagonal entries over which to spread at most three units of spectral weight; the blocks themselves are obtained by scaling the rows of a 3×3 discrete Fourier transform matrix. More generally, UNTFs with redundancy greater than $\frac{J}{J-1}$ can be constructed using $J \times J$ submatrices. Note that these lower levels of redundancy are only bought at the expense of a loss in sparsity, and in particular, a loss of orthogonality relations between the frame elements themselves. We have focused on the use of 2×2 submatrices since it is precisely these orthogonality relations which facilitate our Gabor TFF construction. In particular, by playing Spectral Tetris with only 1×1 and 2×2 submatrices, that is, by repeatedly applying the rules of Theorem 8, one obtains a UNTF in which many frame elements are mutually orthogonal:

Theorem 9. *For any $M, N \in \mathbb{N}$ such that $2N \le M$, there exists a unit norm tight frame $\{f_m\}_{m=1}^M$ for $\mathbb{C}^N$ with the property that $\langle f_m, f_{m'} \rangle = 0$ whenever $m' - m \ge \lfloor \frac{M}{N} \rfloor + 3$.*

When $M = 11$, $N = 4$, the previous result states that the UNTF of Example 6 satisfies $\langle f_m, f_{m'} \rangle = 0$ whenever $m' - m \ge 5$. Moreover, since $\langle f_7, f_3 \rangle \ne 0$, we see that Theorem 9's condition on $m' - m$ is, in fact, the best possible.

Also note that although this section's results were given in the context of complex Euclidean space for the sake of consistency, the frames obtained by playing Spectral Tetris with 1×1 and 2×2 submatrices are, in fact, real-valued. We believe the simplicity of this construction rivals that of real harmonic frames, consisting of samples of sines and cosines. In particular, Spectral Tetris provides a very simple proof of the existence of real UNTFs for any $M \ge N$: when $2N \le M$, the construction is direct; Naimark complements then give real UNTFs with redundancy less than two.

3.1 Gabor fusion frames

We conclude this paper by providing the second half of a general method for constructing (K, L, N)-TFFs when $K \ge \lceil \frac{N}{L} \rceil + 2$. The key idea is to modulate UNTFs whose frame elements satisfy certain orthogonality relations, such as those provided by Theorem 9:

Theorem 10. *If $\{f_n\}_{n=1}^N$ is a UNTF for $\mathbb{C}^L$ and $\langle f_n, f_{n'} \rangle = 0$ whenever K divides $n' - n \ne 0$, then $\{g_{k,l}\}_{k=1,\, l=1}^{K,\ L} \subseteq \mathbb{C}^N$*

$$g_{k,l}(n) = \frac{\sqrt{L}}{\sqrt{N}}\, \mathrm{e}^{2\pi \mathrm{i}(k-1)n/K} f_n(l)$$

generates a (K, L, N)-TFF for $\mathbb{C}^N$.

We note that the frame vectors produced by Theorem 10 are not modulates of the original frame vectors themselves, but rather their coordinate vectors. That is, the analysis operator of $\{g_{k,l}\}_{k=1,\, l=1}^{K,\ L}$ is obtained by stacking modulated copies of the synthesis operator of $\{f_n\}_{n=1}^N$, as illustrated in the following example. As seen in Table 1, this results in a collection of vectors which, from afar, appear as translates and modulates of a single function, and as such are dubbed *Gabor fusion frames*.

Example 11. Recall that, by Theorem 9, the UNTF $\{f_m\}_{m=1}^{11}$ for $\mathbb{C}^4$ constructed in Example 6 satisfies $\langle f_m, f_{m'} \rangle = 0$ whenever $m' - m \ge 5$. Applying Theorem 10 to this UNTF with $K = 5$ produces a $(5, 4, 11)$-TFF, as given in Table 1.

Applying this idea in general, that is, using Theorem 10 to modulate the Spectral Tetris constructions of Theorem 9, yields (K, L, N)-TFFs for all $K, L, N \in \mathbb{N}$ such that $2L < N$, L does not divide N and $K \ge \lfloor \frac{N}{L} \rfloor + 3 = \lceil \frac{N}{L} \rceil + 2$. That is, combining Theorems 9 and 10 yields Theorem 5.

Table 1. The analysis operator of a $(5,4,11)$-TFF, as described in Example 11. Here, $w := \mathrm{e}^{-2\pi\mathrm{i}/5}$. The rows of this matrix form a TFF for $\mathbb{C}^{11}$ consisting of 5 subspaces, each of dimension 4. Here, a given pair of rows belong to the same subspace if their indices differ by a multiple of 5.

$$\begin{bmatrix}
1 & 1 & \frac{\sqrt{3}}{\sqrt{8}} & \frac{\sqrt{3}}{\sqrt{8}} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & w & \frac{\sqrt{3}}{\sqrt{8}}w^2 & \frac{\sqrt{3}}{\sqrt{8}}w^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & w^2 & \frac{\sqrt{3}}{\sqrt{8}}w^4 & \frac{\sqrt{3}}{\sqrt{8}}w^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & w^3 & \frac{\sqrt{3}}{\sqrt{8}}w & \frac{\sqrt{3}}{\sqrt{8}}w^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & w^4 & \frac{\sqrt{3}}{\sqrt{8}}w^3 & \frac{\sqrt{3}}{\sqrt{8}}w & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{\sqrt{5}}{\sqrt{8}} & -\frac{\sqrt{5}}{\sqrt{8}} & 1 & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}} & 0 & 0 & 0 & 0\\
0 & 0 & \frac{\sqrt{5}}{\sqrt{8}}w^2 & -\frac{\sqrt{5}}{\sqrt{8}}w^3 & w^4 & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}}w & 0 & 0 & 0 & 0\\
0 & 0 & \frac{\sqrt{5}}{\sqrt{8}}w^4 & -\frac{\sqrt{5}}{\sqrt{8}}w & w^3 & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}}w^2 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{\sqrt{5}}{\sqrt{8}}w & -\frac{\sqrt{5}}{\sqrt{8}}w^4 & w^2 & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}}w^3 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{\sqrt{5}}{\sqrt{8}}w^3 & -\frac{\sqrt{5}}{\sqrt{8}}w^2 & w & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}}w^4 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}} & 1 & \frac{\sqrt{7}}{\sqrt{8}} & \frac{\sqrt{7}}{\sqrt{8}} & 0\\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}}w & w^2 & \frac{\sqrt{7}}{\sqrt{8}}w^3 & \frac{\sqrt{7}}{\sqrt{8}}w^4 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}}w^2 & w^4 & \frac{\sqrt{7}}{\sqrt{8}}w & \frac{\sqrt{7}}{\sqrt{8}}w^3 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}}w^3 & w & \frac{\sqrt{7}}{\sqrt{8}}w^4 & \frac{\sqrt{7}}{\sqrt{8}}w^2 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}}w^4 & w^3 & \frac{\sqrt{7}}{\sqrt{8}}w^2 & \frac{\sqrt{7}}{\sqrt{8}}w & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{\sqrt{8}} & -\frac{\sqrt{7}}{\sqrt{8}} & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{\sqrt{8}}w^3 & -\frac{\sqrt{7}}{\sqrt{8}}w^4 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{\sqrt{8}}w & -\frac{\sqrt{7}}{\sqrt{8}}w^3 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{\sqrt{8}}w^4 & -\frac{\sqrt{7}}{\sqrt{8}}w^2 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{\sqrt{8}}w^2 & -\frac{\sqrt{7}}{\sqrt{8}}w & 1
\end{bmatrix}$$

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