Constructing tight fusion frames

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Abstract

Tight fusion frames are an emerging concept of frame theory with applications in distributed processing and communications. However, very little has been determined about the existence of such frames. We completely resolve the question of existence in the special case where the underlying space is finite-dimensional and the fusion frame's subspaces have equal dimension. That is, we precisely determine the conditions under which there exists a set of equal-rank orthogonal projection matrices whose sum is a scalar multiple of the identity matrix. The characterizing set of requirements is very mild, and as such, these frames often exist. Our methods are completely constructive, relying on a new, flexible and elementary method for constructing unit norm tight frames.

Keywords: tight, fusion, frames

1. Introduction

A *tight fusion frame* (TFF) is a sequence of orthogonal projection operators that sum to a scalar multiple of the identity operator. Such frames were introduced in [4], and later refined in [6]. TFFs are robust against additive noise and erasures [2, 5, 8], and as such, are well-suited for emerging real-world applications in communications and distributed sensing [7, 10, 11]. In particular, [2] shows that a TFF is maximally robust against the loss of a single projection precisely when the TFF's projection operators have equal rank; we focus exclusively on this special case. To be precise, a sequence $\{P_k\}_{k=1}^K$ of $N \times N$ orthogonal projection matrices of rank L is a (K, L, N)-TFF if there exists $A > 0$ such that:

$$A\mathrm{I} = \sum_{k=1}^{K} P_k. \tag{1}$$

In this paper, we completely characterize the triples (K, L, N) for which a corresponding TFF exists. Characterizing the existence of such frames has proven difficult; frame potential arguments [3, 9] have shown that for any fixed $\alpha > 1$ and L, there exists an index $N_0 = N_0(\alpha, L)$ such that (K, L, N)-TFFs will exist whenever $N \geq N_0$ and $K \geq \alpha N$. Our work below improves upon these sufficient conditions, showing that, in truth, K only needs to be a little larger than $\frac{N}{L}$. To be precise, our first main result is the following partial characterization:

Theorem 1. *Let $K, L, N \in \mathbb{N}$ satisfy $L \leq N$. If L divides N, then (K, L, N)-TFFs exist if and only if $K \geq \frac{N}{L}$. If L does not divide N and we further assume that $2L < N$, then:*

i. *If (K, L, N)-TFFs exist, then $K \geq \lceil \frac{N}{L} \rceil + 1$.*
ii. *If $K \geq \lceil \frac{N}{L} \rceil + 2$, then (K, L, N)-TFFs exist.*

The proof of this result is entirely constructive in the cases where TFFs exist. Next, to fully characterize the existence of equal-rank TFFs, we employ two distinct methods of taking orthogonal complements of a TFF. This characterization is given in our second main result:

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Theorem 2. *For each $K, L, N \in \mathbb{N}$ such that $L < N$, the existence of (K, L, N)-TFFs can be completely resolved using Theorem 1 along with at most one application of the fact that:*

(K, L, N)*-TFFs exist if and only if* $(K, N - L, N)$*-TFFs exist, provided* $L < N$,

and at most $L - 1$ repeated applications of the fact that:

(K, L, N)*-TFFs exist if and only if* $(K, (K-1)L - N, KL - N)$*-TFFs exist, provided* $L < KL - N$.

In the next section, we discuss how TFFs can be regarded as special cases of unit norm tight frames, a basic idea which underlies nearly all of our arguments. Using this idea, we then introduce several basic methods for constructing new TFFs from existing ones. These constructions employ either tensor products or orthogonal complements. In Section 3, we introduce a new fundamental technique for constructing unit norm tight frames. This method resembles the popular game Tetris™, as it involves building a flat spectrum with blocks of fixed area. In the fourth section, this *Spectral Tetris* construction is then combined with a new, modulation-based method for building TFFs, yielding *modulated fusion frames*, whose existence is the key to proving Theorem 1. In the final section, we combine our results with some new analysis to prove Theorem 2. To be precise, we provide a simple iterative algorithm, dubbed the Tight Fusion Frame Existence Test, that quickly resolves the existence of equal-rank TFFs in the few cases where Theorem 1 is ambiguous.

2. Basic constructions

The *synthesis operator* of a finite sequence of vectors $\{f_m\}_{m=1}^M$ in $\mathbb{C}^N$ is $F : \mathbb{C}^M \to \mathbb{C}^N$,

$$Fg = \sum_{m=1}^{M} g(m) f_m,$$

where, here and throughout, "$g(m)$" denotes the mth entry of the vector g. That is, F is an $N \times M$ matrix whose mth column is f_m. Generally speaking, *frame theory* is the study of how $\{f_m\}_{m=1}^M$ should be chosen so as to ensure that the corresponding *frame operator* FF^* is well-conditioned. In particular, $\{f_m\}_{m=1}^M$ is a *tight frame* if there exists $A > 0$ such that $FF^* = A\mathrm{I}$, namely that:

$$Af = \sum_{m=1}^{M} \langle f, f_m \rangle f_m \tag{2}$$

for all $f \in \mathbb{C}^N$, or equivalently, that:

$$\sum_{m=1}^{M} f_m(n)\overline{f_m(n')} = \begin{cases} A, & n = n', \\ 0, & n \neq n'. \end{cases}$$

A *unit norm tight frame* (UNTF) is a tight frame $\{f_m\}_{m=1}^M$ which further satisfies $\|f_m\| = 1$ for all $m = 1, \dots, M$. UNTFs are known to exist for any $M \geq N$; the standard example is the *harmonic frame*, whose synthesis operator is obtained by extracting any N distinct rows from a suitably scaled $M \times M$ discrete Fourier transform matrix. UNTFs provide Parseval-like decompositions in terms of nonorthogonal vectors of unit norm.

Fusion frame theory generalizes these concepts. In particular, when each f_m is of unit norm, the summands of (2), namely, the operators $f \mapsto \langle f, f_m \rangle f_m$, are rank-one orthogonal projections. Fusion frame theory is the study of sums of projections of arbitrary rank, leading to the definition of a tight fusion frame given in (1). In particular, recall that $\{P_k\}_{k=1}^K$ is a (K, L, N)-TFF if each P_k is an $N \times N$ orthogonal projection matrix of rank L. Letting $\{f_{k,l}\}_{l=1}^L$ be an orthonormal basis for the range of P_k, we classically know that:

$$P_k f = \sum_{l=1}^{L} \langle f, f_{k,l} \rangle f_{k,l}$$

for all $f \in \mathbb{C}^N$. Summing these equations over $k = 1, \dots, K$ yields:

$$\sum_{k=1}^{K} P_k f = \sum_{k=1}^{K} \sum_{l=1}^{L} \langle f, f_{k,l} \rangle f_{k,l},$$

a fact which, in light of (1) and (2), shows that every equal-rank TFF arises from a traditional tight frame that satisfies additional orthogonality requirements. To be precise:

Definition 3. A sequence $\{f_{k,l}\}_{k=1,\,l=1}^{K\ \ \ L} \subset \mathbb{C}^N$ *generates* a (K, L, N)-TFF if:

i. $\{f_{k,l}\}_{l=1}^{L}$ is orthonormal for every $k = 1, \ldots, K$.
ii. $\{f_{k,l}\}_{k=1,\,l=1}^{K\ \ \ L}$ is a tight frame for $\mathbb{C}^N$, that is, there exists $A > 0$ such that for any $f \in \mathbb{C}^N$,

$$Af = \sum_{k=1}^{K}\sum_{l=1}^{L}\langle f, f_{k,l}\rangle f_{k,l}. \tag{3}$$

Equivalently, the rows of the synthesis operator are mutually orthogonal with equal norm:

$$\sum_{k=1}^{K}\sum_{l=1}^{L} f_{k,l}(n)\overline{f_{k,l}(n')} = \begin{cases} A, & n = n', \\ 0, & n \neq n'. \end{cases} \tag{4}$$

From this perspective, we see that (K, L, N)-TFFs are actually special cases of UNTFs of KL elements for $\mathbb{C}^N$, and as such, the tight frame constant in (3) is necessarily $A = \frac{KL}{N}$, where $KL \geq N$, see [1]. We now exploit this UNTF-based representation, providing several methods for constructing new TFFs from existing ones.

2.1. Tensor products

Inner products distribute multiplicatively over Kronecker tensor products. As such, the tensor product of two TFFs is another TFF:

Theorem 4. *If* $\{f_{k_1,l_1}\}_{k_1=1,\,l_1=1}^{K_1\ \ \ L_1}$ *and* $\{g_{k_2,l_2}\}_{k_2=1,\,l_2=1}^{K_2\ \ \ L_2}$ *generate* (K_1, L_1, N_1)*- and* (K_2, L_2, N_2)*-TFFs respectively, then:*

$$\{h_{k,l}\}_{k=1,\ \ l=1}^{K_1K_2\,L_1L_2}, \qquad h_{k,l}(n) := f_{k_1,l_1}(n_1)g_{k_2,l_2}(n_2)$$

generates a (K_1K_2, L_1L_2, N_1N_2)*-TFF, where* $k = (k_1 - 1)K_2 + k_2$*,* $l = (l_1 - 1)L_2 + l_2$*, and* $n = (n_1 - 1)N_2 + n_2$*.*

Proof. We use (4) to show that $\{h_{k,l}\}_{k=1,\ \ l=1}^{K_1K_2\,L_1L_2}$ is tight; writing any $n, n' \in [1, N_1N_2]$ uniquely in terms of $n_1, n_1' \in [1, N_1]$ and $n_2, n_2' \in [1, N_2]$ as given in the statement of the result, one easily finds that:

$$\sum_{k=1}^{K_1K_2}\sum_{l=1}^{L_1L_2} h_{k,l}(n)\overline{h_{k,l}(n')} = \sum_{k_1=1}^{K_1}\sum_{l_1=1}^{L_1} f_{k_1,l_1}(n_1)\overline{f_{k_1',l_1'}(n_1')}\sum_{k_2=1}^{K_2}\sum_{l_2=1}^{L_2} f_{k_2,l_2}(n_2)\overline{f_{k_2',l_2'}(n_2')} = \begin{cases} \frac{K_1K_2L_1L_2}{N_1N_2}, & n = n', \\ 0, & n \neq n'. \end{cases}$$

Similarly, writing k, k', l, l' in terms of $k_1, k_1', k_2, k_2', l_1, l_1', l_2, l_2'$, one may easily show that:

$$\langle h_{k,l}, h_{k',l'}\rangle = \langle f_{k_1,l_1}, f_{k_1',l_1'}\rangle\langle g_{k_2,l_2}, g_{k_2',l_2'}\rangle. \tag{5}$$

Letting $k = k'$ and $l = l'$ in (5) gives $\|h_{k,l}\| = 1$, and so $\{h_{k,l}\}_{k=1,\ \ l=1}^{K_1K_2\,L_1L_2}$ is, in fact, a UNTF. Moreover, if $k = k'$ but $l \neq l'$, then either $l_1 \neq l_1'$ or $l_2 \neq l_2'$, and so (5) gives $\langle h_{k,l}, h_{k,l'}\rangle = 0$, implying that for any fixed $k = 1, \ldots, K_1K_2$, the subcollection $\{h_{k,l}\}_{l=1}^{L_1L_2}$ is orthonormal. □

Though elementary, this tensor product construction provides a simple proof of the first part of Theorem 1:

Corollary 5. *If* $K, L, N \in \mathbb{N}$ *and* L *divides* N*, then* (K, L, N)*-TFFs exist if and only if* $K \geq \frac{N}{L}$*.*

Proof. ($\Rightarrow$) If a (K, L, N)-TFF exists, then any sequence that generates it consists of KL vectors that span $\mathbb{C}^N$, implying $KL \geq N$. ($\Leftarrow$) If $K \geq \frac{N}{L}$, then there exists a UNTF of K elements for $\mathbb{C}^{\frac{N}{L}}$, see [1]. That is, $(K, 1, \frac{N}{L})$-TFFs exist. Also, any orthonormal basis for $\mathbb{C}^L$ is a $(1, L, L)$-TFF. By Theorem 4, the tensor product of these two sequences generates a $(K \cdot 1, 1 \cdot L, \frac{N}{L} \cdot L) = (K, L, N)$-TFF. □

2.2. *Complementary fusion frames*

In this subsection, we consider two distinct orthogonal complements of a TFF. For the first complement, let $\{f_{k,l}\}_{k=1,\,l=1}^{K\ \ \ L}$ generate a (K, L, N)-TFF, and for each $k = 1, \ldots, K$, extend the orthonormal sequence $\{f_{k,l}\}_{l=1}^{L}$ to an orthonormal basis for $\mathbb{C}^N$. We claim that the vectors from this extension generate another TFF, dubbed the *spatial complement* of the original. This new TFF possesses the same number of subspaces as the original, and the dimension of the underlying space remains the same—only the dimension of the subspaces changes:

Theorem 6. *If $\{f_{k,l}\}_{k=1,\,l=1}^{K\ \ \ L}$ generates a (K, L, N)-TFF and $L < N$, then any $\{g_{k,l'}\}_{k=1,\,l'=1}^{K\ \ \ N-L} \subset \mathbb{C}^N$ such that:*

$$\textit{for each } k = 1, \ldots, K, \{f_{k,l}\}_{l=1}^{L} \cup \{g_{k,l'}\}_{l'=1}^{N-L} \textit{ is an orthonormal basis for } \mathbb{C}^N,$$

generates a $(K, N-L, N)$-TFF.

Proof. For any $k = 1, \ldots, K$, the sequence $\{f_{k,l}\}_{l=1}^{L} \cup \{g_{k,l'}\}_{l'=1}^{N-L}$ is an orthonormal basis for $\mathbb{C}^N$, implying $\{g_{k,l'}\}_{k=1,\,l'=1}^{K\ \ \ N-L}$ is orthonormal and moreover:

$$f = \sum_{l=1}^{L} \langle f, f_{k,l} \rangle f_{k,l} + \sum_{l'=1}^{N-L} \langle f, g_{k,l'} \rangle g_{k,l'}$$

for all $f \in \mathbb{C}^N$. Since $\{f_{k,l}\}_{k=1,\,l=1}^{K\ \ \ L}$ generates a (K, L, N)-TFF, summing these equations over $k = 1, \ldots, K$ yields:

$$\sum_{k=1}^{K} \sum_{l'=1}^{N-L} \langle f, g_{k,l'} \rangle g_{k,l'} = Kf - \sum_{k=1}^{K} \sum_{l=1}^{L} \langle f, f_{k,l} \rangle f_{k,l} = Kf - \tfrac{KL}{N} f = \tfrac{K(N-L)}{N} f$$

for all $f \in \mathbb{C}^N$, as claimed. □

A second way to take an orthogonal complement of a TFF is to extend the $N \times KL$ synthesis matrix to a $KL \times KL$ unitary matrix, and then consider the $(N - KL) \times KL$ extension. We claim these new vectors also generate a TFF, termed the *Naimark complement* of the original, as the construction makes use of Naimark's argument that every 1-tight frame is the projection of an orthonormal basis. Here, the number and dimension of the new TFF's subspaces are equal to those of the original, but the dimension of the underlying space changes:

Theorem 7. *If $\{f_{k,l}\}_{k=1,\,l=1}^{K\ \ \ L}$ generates a (K, L, N)-TFF and $N < KL$, then any $\{g_{k,l}\}_{k=1,\,l=1}^{K\ \ \ L} \subset \mathbb{C}^{KL-N}$ such that:*

$$\{\tfrac{\sqrt{N}}{\sqrt{KL}} f_{k,l} \oplus \tfrac{\sqrt{KL-N}}{\sqrt{KL}} g_{k,l}\}_{k=1,\,l=1}^{K\ \ \ L} \textit{ is an orthonormal basis for } \mathbb{C}^{KL}$$

generates a $(K, L, KL - N)$-TFF.

Proof. Letting $h_{k,l} = \frac{\sqrt{N}}{\sqrt{KL}} f_{k,l} \oplus \frac{\sqrt{KL-N}}{\sqrt{KL}} g_{k,l}$, note that for any $k = 1, \ldots, K$ and any $l, l' = 1, \ldots, L$,

$$\langle h_{k,l}, h_{k,l'} \rangle = \tfrac{N}{KL} \langle f_{k,l}, f_{k,l'} \rangle + \tfrac{KL-N}{KL} \langle g_{k,l}, g_{k,l'} \rangle.$$

When combined with the fact that $\{f_{k,l}\}_{l=1}^{L}$ and $\{h_{k,l}\}_{l=1}^{L}$ are orthonormal, this equation implies that $\{g_{k,l}\}_{l=1}^{L}$ is also orthonormal. At the same time, since $\{h_{k,l}\}_{k=1,\,l=1}^{K\ \ \ L}$ is an orthonormal basis for $\mathbb{C}^{KL}$, then its synthesis operator is unitary. As such, the rows of this matrix are also orthonormal; for $n, n' = 1, \ldots, KL - N$, we have:

$$\tfrac{KL-N}{KL} \sum_{k=1}^{K} \sum_{l=1}^{L} g_{k,l}(n) \overline{g_{k,l}(n')} = \sum_{k=1}^{K} \sum_{l=1}^{L} h_{k,l}(n + KL) \overline{h_{k,l}(n' + KL)} = \begin{cases} 1, & n = n', \\ 0, & n \neq n', \end{cases}$$

and so $\{g_{k,l}\}_{k=1,\,l=1}^{K\ \ \ L}$ satisfies (4). □

Indeed, the relations on (K, L, N) in Theorems 6 and 7 are self-dual, an so we have the following:

Corollary 8. *For each $(K, L, N) \in \mathbb{N}$ such that $L \leq N$,*

i. *(Spatial complements) If $L < N$, then (K, L, N)-TFFs exist if and only if $(K, N - L, N)$-TFFs exist.*

ii. *(Naimark complements) If $N < KL$, then (K, L, N)-TFFs exist if and only if $(K, L, KL - N)$-TFFs exist.*

We have already noted that in order for (K, L, N)-TFFs to exist, one needs $KL \geq N$; we now use Corollary 8 to prove a stronger necessary condition on existence, given in Theorem 1:

Corollary 9. *If (K, L, N)-TFFs exist and L does not divide N, then $K \geq \lceil \frac{N}{L} \rceil + 1$.*

Proof. If (K, L, N)-TFFs exist, then $KL \geq N$. Since L does not divide N, then $KL > N$, and so $(K, L, KL - N)$-TFFs exist by the previous result. Thus, there exists L orthonormal vectors in $\mathbb{C}^{KL-N}$, and as such, $L \leq KL - N$. Simplifying, we find $K \geq \frac{N}{L} + 1$. Since K is an integer, taking the ceiling of both sides of this equation yields the result. □

We note that the necessary condition of Corollary 9 is not sufficient. In particular, $(3, 3, 4)$-TFFs do not exist, despite the fact that $3 \geq \lceil \frac{4}{3} \rceil + 1$. Indeed, if a $(3, 3, 4)$-TFF did exist, then its spatial complement, obtained by applying Corollary 8.i, would be a $(3, 1, 4)$-TFF; such TFFs do not exist by Corollary 9, since $3 < \lceil \frac{4}{1} \rceil + 1$.

One may preclude such simple counterexamples to the sufficiency of Corollary 9's condition by making the further requirement that $2L < N$. However, even in this case, $K \geq \lceil \frac{N}{L} \rceil + 1$ is not sufficient: $(4, 4, 11)$-TFFs do not exist, despite the fact that $4 \geq \lceil \frac{11}{4} \rceil + 1$ and $2(4) < 11$. To be precise, if a $(4, 4, 11)$-TFF did exist, then its Naimark complement, obtained by applying Corollary 8.ii, would be a $(4, 4, 5)$-TFF, whose spatial complement would, in turn, be a $(4, 1, 5)$-TFF; such frames do not exist since $4 < \lceil \frac{5}{1} \rceil + 1$.

To summarize, the conditions $2L < N$ and $K \geq \lceil \frac{N}{L} \rceil + 1$ are not sufficient to guarantee the existence of (K, L, N)-TFFs. However, one of the main results of this paper, as encapsulated in the final statement of Theorem 1, is to show that a very slight strengthening of these conditions is actually sufficient for existence. Specifically, over the course of the next two sections, we will provide an explicit construction of a (K, L, N)-TFF for each $K, L, N \in \mathbb{N}$ such that L does not divide N, $2L < N$ and $K \geq \lceil \frac{N}{L} \rceil + 2$. That is, we will show that TFFs indeed exist whenever the number of subspaces K is at least two more than what is absolutely necessary. Moreover, in the final section, we will show that the existence of equal-rank TFFs is completely resolved using this construction along with a finite number of repeated applications of Corollary 8.

3. Spectral Tetris

In this section, we provide the first half of a general method for constructing (K, L, N)-TFFs when $K \geq \lceil \frac{N}{L} \rceil + 2$. The key idea is to revisit the simpler problem of constructing UNTFs, that is, sequences $\{f_m\}_{m=1}^{M}$ of unit vectors in $\mathbb{C}^N$ that satisfy (2). In brief, we want to construct $N \times M$ synthesis matrices F which have:

i. columns of unit norm,
ii. orthogonal rows, meaning the frame operator FF^* is diagonal,
iii. rows of constant norm, meaning FF^* is a constant multiple of the identity matrix.

Despite a decade of study, very few general constructions of finite-dimensional UNTFs are known. Moreover, these known methods unfortunately manipulate all frame elements simultaneously. In this section, we show that constructing certain examples of UNTFs need not be so difficult. In particular, we provide a new, iterative method for constructing UNTFs, building them one or two vectors at a time. The key idea is to iteratively build a matrix F which, at each iteration, exactly satisfies (i) and (ii), and gets closer to satisfying (iii). We call this method *Spectral Tetris*, as it involves building a flat spectrum out of blocks of fixed area. Here, an illustrative example is helpful:

Example 10. In the previous section, we showed that $(4, 4, 11)$-TFFs did not exist, despite the fact that these K, L and N satisfy the necessary condition for existence given in Corollary 9. At the same time, we claim in Theorem 1 that a slightly stronger requirement, $K \geq \lceil \frac{N}{L} \rceil + 2$, is indeed sufficient for existence, provided L does not divide N and $2L < N$. In particular, Theorem 1 asserts that $(5, 4, 11)$-TFFs exist. In this and the following sections, we will show how to explicitly construct such a TFF, so as to illustrate the simple ideas behind the proof of Theorem 1.ii. The construction is performed over two stages. The first stage, given in the present example, is to play Spectral Tetris, yielding a sparse UNTF of 11 elements for $\mathbb{C}^4$. In the second stage, this UNTF is then modulated to produce a $(5, 4, 11)$-TFF, as described in Example 15.

Our immediate goal is to create a 4×11 matrix F such that $FF^* = \frac{11}{4}\mathrm{I}$. As such, we begin with an arbitrary 4×11 matrix, and let the first two frame elements be copies of the first standard basis element e_1:

$$F = \begin{bmatrix} 1 & 1 & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? & ? & ? & ? & ? & ? \end{bmatrix}. \tag{6}$$

If the remaining unknown entries are chosen so that F has orthogonal rows, then FF^* will be a diagonal matrix. Currently, the diagonal entries of FF^* are mostly unknown, having the form $\{2+?, ?, ?, ?\}$. Also note that if the remainder of the first row of F is set to zero, then the first diagonal entry of FF^* would be $2 < \frac{11}{4}$. Thus, we need to add more weight to this row. However, making the third column of F another copy of e_1 would add too much weight, as $3 > \frac{11}{4}$. Therefore, we need a way to put $\frac{11}{4} - 2 = \frac{3}{4}$ more weight in the first row without compromising the orthogonality of the rows of F nor the normality of its columns. The key idea is to realize that for any $0 \leq x \leq 2$, there exists a 2×2 matrix $T(x)$ with orthogonal rows and unit-length columns such that $T(x)T^*(x)$ is a diagonal matrix with diagonal entries $\{x, 2 - x\}$. Specifically, we have:

$$T(x) := \tfrac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{x} & \sqrt{x} \\ \sqrt{2-x} & -\sqrt{2-x} \end{bmatrix}, \quad T(x)T^*(x) = \begin{bmatrix} x & 0 \\ 0 & 2-x \end{bmatrix}.$$

We define the third and fourth columns of F according to such a matrix $T(x)$, where $x = \frac{11}{4} - 2 = \frac{3}{4}$:

$$F = \begin{bmatrix} 1 & 1 & \frac{\sqrt{3}}{\sqrt{8}} & \frac{\sqrt{3}}{\sqrt{8}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{5}}{\sqrt{8}} & -\frac{\sqrt{5}}{\sqrt{8}} & ? & ? & ? & ? & ? & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? & ? & ? & ? & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? & ? & ? & ? & ? & ? \end{bmatrix}. \tag{7}$$

The diagonal entries of FF^* are now $\{\frac{11}{4}, \frac{5}{4}+?, ?, ?\}$. The first row now has sufficient weight, and so its remaining entries are set to zero. The second entry is currently falling short by $\frac{11}{4} - \frac{5}{4} = \frac{6}{4} = 1 + \frac{2}{4}$, and as such, we make the fifth column e_2, while the sixth and seventh arise from $T(\frac{2}{4})$:

$$F = \begin{bmatrix} 1 & 1 & \frac{\sqrt{3}}{\sqrt{8}} & \frac{\sqrt{3}}{\sqrt{8}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{5}}{\sqrt{8}} & -\frac{\sqrt{5}}{\sqrt{8}} & 1 & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}} & ? & ? & ? & ? \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & ? & ? & ? & ? \end{bmatrix}. \tag{8}$$

The diagonal entries of FF^* are now $\{\frac{11}{4}, \frac{11}{4}, \frac{6}{4}+?, ?\}$, where the third diagonal entry is falling short by $\frac{11}{4} - \frac{6}{4} = \frac{5}{4} = 1 + \frac{1}{4}$. We therefore take the eighth column of F as e_3, let the ninth and tenth columns arise from $T(\frac{1}{4})$, and make the final column be e_4, yielding the desired UNTF:

$$F = \begin{bmatrix} 1 & 1 & \frac{\sqrt{3}}{\sqrt{8}} & \frac{\sqrt{3}}{\sqrt{8}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{5}}{\sqrt{8}} & -\frac{\sqrt{5}}{\sqrt{8}} & 1 & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}} & 1 & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{\sqrt{8}} & -\frac{\sqrt{7}}{\sqrt{8}} & 1 \end{bmatrix}. \tag{9}$$

In this construction, column vectors are either introduced one at a time, such as $\{f_1\}$, $\{f_2\}$, $\{f_5\}$, $\{f_8\}$ or $\{f_{11}\}$, or in pairs, such as $\{f_3, f_4\}$, $\{f_6, f_7\}$ or $\{f_9, f_{10}\}$. Each singleton contributes a value of 1 to a particular diagonal entry of FF^*, while each pair spreads two units of weight over two entries. Overall, we have formed a flat spectrum, $\{\frac{11}{4}, \frac{11}{4}, \frac{11}{4}, \frac{11}{4}\}$, from blocks of area 1 or 2. This construction is reminiscent of the game Tetris, as illustrated in Figure 1.

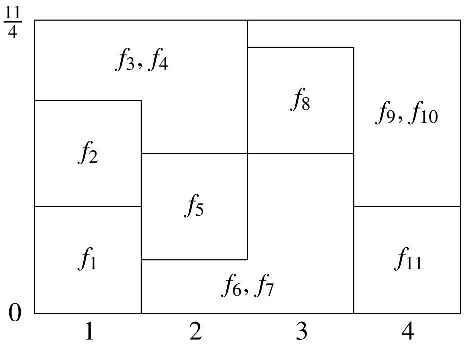

Figure 1: The Spectral Tetris construction of a UNTF of 11 elements for $\mathbb{C}^4$, as detailed in Example 10. Each of the four columns corresponds to a diagonal entry of the frame operator FF^*, and each block represents the contribution made to these entries by the corresponding frame elements. For example, the single frame element $\{f_2\}$ contributes $\{1,0,0,0\}$ to the diagonal, while the pair $\{f_6, f_7\}$ contributes $\{0, \frac{2}{4}, \frac{6}{4}, 0\}$. The area of the blocks is determined by the number of frame elements that generate them: blocks that arise from a single element have unit area, while blocks that arise from two elements have an area of 2. In order for $\{f_m\}_{m=1}^{11}$ to be a UNTF for $\mathbb{C}^4$, these blocks needed to stack to a uniform height of $\frac{11}{4}$. By building a rectangle from blocks of given areas, we are essentially playing Tetris with the spectrum of FF^*.

We conclude this example by pointing out a crucial consequence of this Spectral Tetris construction: the frame vectors are extremely sparse, with many pairs of vectors having mutually disjoint support. In particular, we have that f_m and $f_{m'}$ are orthogonal whenever $m - m' \geq 5$. More generally, we shall show that whenever Spectral Tetris is played to form a UNTF, the resulting frame elements satisfy $\langle f_m, f_{m'}\rangle = 0$ whenever $m' - m \geq \lfloor \frac{M}{N} \rfloor + 3$. These orthogonality relations will play a critical role in the next section, where Spectral Tetris UNTFs will be modulated to form modulated TFFs.

In order to formalize the Spectral Tetris argument used in the previous example, we introduce the following notion:

Definition 11. We say that a sequence $\{f_m\}_{m=1}^M$ is an (m_0, n_0)-*proto unit norm tight frame* (PUNTF) for $\mathbb{C}^N$ if:

i. $\displaystyle\sum_{n=1}^{N} |f_m(n)|^2 = \begin{cases} 1, & m \leq m_0, \\ 0, & m > m_0, \end{cases}$

ii. $\displaystyle\sum_{m=1}^{M} f_m(n)\overline{f_m(n')} = 0$ for all $n, n' = 1, \dots, N, n \neq n'$,

iii. $\displaystyle\sum_{m=1}^{M} |f_m(n)|^2 = \begin{cases} \frac{M}{N}, & n < n_0, \\ 0, & n > n_0, \end{cases}$

iv. $\displaystyle 1 \leq \sum_{m=1}^{M} |f_m(n_0)|^2 \leq \frac{M}{N}$.

That is, $\{f_m\}_{m=1}^M$ is an (m_0, n_0)-PUNTF for $\mathbb{C}^N$ precisely when its $N \times M$ synthesis matrix F vanishes off of its upper-left $n_0 \times m_0$ submatrix, its nonzero columns have unit norm, and its frame operator FF^* is diagonal, with the first $n_0 - 1$ diagonal entries being $\frac{M}{N}$, the n_0th entry lying in $[1, \frac{M}{N}]$, and the remaining entries being zero. In particular, setting "?" entries to zero in (6), (7), (8) and (9) results in $(2,1)$-, $(4,2)$-, $(7,3)$- and $(11,4)$-PUNTFs, respectively. As seen in Example 10, the goal of Spectral Tetris is to iteratively create larger PUNTFs from existing ones, continuing until $(m_0, n_0) = (M, N)$, at which point the PUNTF is a UNTF. We now give the precise rules for enlarging a given PUNTF; here, as in the preceding example, $\{e_n\}_{n=1}^N$ is the standard basis of $\mathbb{C}^N$:

Theorem 12. *Let $2N \leq M$, let $\{f_m\}_{m=1}^M$ be an (m_0, n_0)-PUNTF for $\mathbb{C}^N$, and let* $\lambda := \displaystyle\sum_{m=1}^{M} |f_m(n_0)|^2$.

i. *If* $\lambda \leq \frac{M}{N} - 1$*, then* $m_0 < M$ *and* $\{g_m\}_{m=1}^{M}$,

$$g_m := \begin{cases} f_m, & m \leq m_0, \\ e_{n_0}, & m = m_0 + 1, \\ 0, & m > m_0 + 1, \end{cases}$$

is an $(m_0 + 1, n_0)$*-PUNTF for* $\mathbb{C}^N$.

ii. *If* $\frac{M}{N} - 1 < \lambda < \frac{M}{N}$*, then* $m_0 < M - 2$*,* $n_0 < N$ *and* $\{g_m\}_{m=1}^{M}$,

$$g_m := \begin{cases} f_m, & m \leq m_0, \\ \sqrt{\frac{1}{2}(\frac{M}{N} - \lambda)}e_{n_0} + \sqrt{1 - \frac{1}{2}(\frac{M}{N} - \lambda)}e_{n_0+1}, & m = m_0 + 1, \\ \sqrt{\frac{1}{2}(\frac{M}{N} - \lambda)}e_{n_0} - \sqrt{1 - \frac{1}{2}(\frac{M}{N} - \lambda)}e_{n_0+1}, & m = m_0 + 2, \\ 0, & m > m_0 + 2, \end{cases}$$

is an $(m_0 + 2, n_0 + 1)$*-PUNTF for* $\mathbb{C}^N$.

iii. *If* $\lambda = \frac{M}{N}$ *and* $n_0 < N$*, then* $m_0 < M$ *and* $\{g_m\}_{m=1}^{M}$,

$$g_m := \begin{cases} f_m, & m \leq m_0, \\ e_{n_0+1}, & m = m_0 + 1, \\ 0, & m > m_0 + 1, \end{cases}$$

is an $(m_0 + 1, n_0 + 1)$*-PUNTF for* $\mathbb{C}^N$.

iv. *If* $\lambda = \frac{M}{N}$ *and* $n_0 = N$*, then* $\{f_m\}_{m=1}^{M}$ *is a UNTF for* $\mathbb{C}^N$.

Proof. We first determine a relationship between m_0, n_0 and λ. In particular, the square of the Hilbert-Schmidt norm of the synthesis operator of the (m_0, n_0)-PUNTF $\{f_m\}_{m=1}^{M}$ is:

$$\sum_{m=1}^{M}\sum_{n=1}^{N} |f_m(n)|^2 = \sum_{m=1}^{m_0}\sum_{n=1}^{N} |f_m(n)|^2 + \sum_{m=m_0+1}^{M}\sum_{n=1}^{N} |f_m(n)|^2 = \sum_{m=1}^{m_0} 1 + \sum_{m=m_0+1}^{M} 0 = m_0. \tag{10}$$

We may alternatively evaluate this sum by interchanging summations:

$$\sum_{n=1}^{N}\sum_{m=1}^{M} |f_m(n)|^2 = \sum_{n=1}^{n_0-1}\sum_{m=1}^{M} |f_m(n)|^2 + \sum_{m=1}^{M} |f_m(n_0)|^2 + \sum_{n=n_0+1}^{N}\sum_{m=1}^{M} |f_m(n)|^2 = \sum_{n=1}^{n_0-1} \frac{M}{N} + \lambda + \sum_{n=n_0+1}^{N} 0 = (n_0 - 1)\frac{M}{N} + \lambda. \tag{11}$$

Equating (10) and (11) then gives:

$$\lambda = m_0 - n_0\frac{M}{N} + \frac{M}{N}. \tag{12}$$

Having (12), we turn to proving (i), (ii), (iii) and (iv).

We focus on (ii), as it is the least trivial. In particular, if $\frac{M}{N} - 1 < \lambda < \frac{M}{N}$, then (12) gives:

$$0 < n_0\frac{M}{N} - m_0 < 1. \tag{13}$$

If $n_0 = N$, then (13) implies $0 < M - m_0 < 1$, a contradiction of the fact that $M, m_0 \in \mathbb{N}$. Thus, $n_0 < N$, as claimed. Moreover, substituting the fact that $n_0 \leq N - 1$ into the left-hand inequality of (13) gives:

$$0 < n_0\frac{M}{N} - m_0 \leq (N - 1)\frac{M}{N} - m_0 = M - \frac{M}{N} - m_0 \leq M - 2 - m_0,$$

where the last inequality follows from the global assumption that $2N \leq M$. Thus, $m_0 < M - 2$, as claimed. To continue, note that since $m_0 < M - 2$ and $n_0 < N$, then $m_0 + 1 < M$, $m_0 + 2 < M$ and $n_0 + 1 \leq N$, and so the sequence $\{g_m\}_{m=1}^{M}$ given in (ii) is well-defined. We now verify that $\{g_m\}_{m=1}^{M}$ indeed satisfies the four properties of an

(m_0+2, n_0+1)-PUNTF. Indeed, since $\{f_m\}_{m=1}^M$ is an (m_0,n_0)-PUNTF and since $f_m = g_m$ for all $m \neq m_0+1$ and $m \neq m_0+2$, then Definition 11.i must only be verified for $m = m_0+1$ and $m = m_0+2$:

$$\sum_{n=1}^N |f_{m_0+1}(n)|^2 = \tfrac{1}{2}(\tfrac{M}{N}-\lambda) + [1-\tfrac{1}{2}(\tfrac{M}{N}-\lambda)] = 1 = \tfrac{1}{2}(\tfrac{M}{N}-\lambda) + [1-\tfrac{1}{2}(\tfrac{M}{N}-\lambda)] = \sum_{n=1}^N |f_{m_0+2}(n)|^2.$$

Next, since $\{f_m\}_{m=1}^M$ is an (m_0,n_0)-PUNTF, then we already know that $\{g_m\}_{m=1}^M$ satisfies Definition 11.ii for any distinct n, n' not equal to either n_0 or n_0+1. Definition 11.ii is also immediately satisfied in the case where $n > n_0+1$, as $f_m(n) = 0$ for all $m = 1,\dots,M$, as well as in the case where $n = 1,\dots,n_0-1$ and $n' = n_0+1$, as the supports of the corresponding row vectors are disjoint. The two cases that remain are when $n = 1,\dots,n_0-1$, and $n' = n_0$, in which:

$$\sum_{m=1}^M g_m(n)\overline{g_m(n_0)} = \sum_{m=1}^{m_0+2} g_m(n)\overline{g_m(n_0)} = \sum_{m=1}^{m_0} f_m(n)\overline{f_m(n_0)} + \sum_{m=m_0+1}^{m_0+2} 0\cdot\overline{g_m(n_0)} = 0+0 = 0,$$

and the case $n = n_0$ and $n' = n_0+1$, in which:

$$\sum_{m=1}^M g_m(n_0)\overline{g_m(n_0+1)} = \sum_{m=1}^{m_0} g_m(n_0)\cdot\overline{0} + \sqrt{\tfrac{1}{2}(\tfrac{M}{N}-\lambda)}\sqrt{1-\tfrac{1}{2}(\tfrac{M}{N}-\lambda)} - \sqrt{\tfrac{1}{2}(\tfrac{M}{N}-\lambda)}\sqrt{1-\tfrac{1}{2}(\tfrac{M}{N}-\lambda)} = 0.$$

We next show that $\{g_m\}_{m=1}^M$ satisfies Definition 11.iii in the case where "n_0" is n_0+1. For $n < n_0$ or $n > n_0+1$, this follows immediately from the fact that $\{f_m\}_{m=1}^M$ is an (m_0,n_0)-PUNTF. Else, when $n = n_0$, we have:

$$\sum_{m=1}^M |g_m(n_0)|^2 = \sum_{m=1}^{m_0} |f_m(n_0)|^2 + \tfrac{1}{2}(\tfrac{M}{N}-\lambda) + \tfrac{1}{2}(\tfrac{M}{N}-\lambda) = \lambda + \tfrac{M}{N} - \lambda = \tfrac{M}{N},$$

as needed. Finally, we verify that $\{g_m\}_{m=1}^M$ satisfies Definition 11.iv where "n_0" is n_0+1. Indeed, since:

$$\lambda^* := \sum_{m=1}^M |g_m(n_0+1)|^2 = [1-\tfrac{1}{2}(\tfrac{M}{N}-\lambda)] + [1-\tfrac{1}{2}(\tfrac{M}{N}-\lambda)] = 2 - (\tfrac{M}{N}-\lambda),$$

the assumption that $\frac{M}{N} - 1 < \lambda < \frac{M}{N}$ implies that $1 < \lambda^* < 2$. In particular, since $2N \leq M$, then $1 \leq \lambda^* \leq \frac{M}{N}$, as needed.

Having proven (ii), we return to (i), noting that since $\lambda \leq \frac{M}{N} - 1$, then (12) gives $m_0 - n_0\frac{M}{N} + \frac{M}{N} \leq \frac{M}{N} - 1$, and so:

$$m_0 \leq n_0\tfrac{M}{N} - 1 \leq N\tfrac{M}{N} - 1 = M - 1 < M.$$

That is, $m_0 < M$ as claimed, and as such, $\{g_m\}_{m=1}^M$ is well-defined. The proof of the fact that $\{g_m\}_{m=1}^M$ is a UNTF is very similar to, but simpler than, the parallel argument above in the proof of (ii), and as such, is omitted. Similarly, to show (iii), note that (12) gives $\frac{M}{N} = \lambda = m_0 - n_0\frac{M}{N} + \frac{M}{N}$, and thus $m_0 = n_0\frac{M}{N} < N\frac{M}{N} = M$, as claimed. Therefore, $\{g_m\}_{m=1}^M$ is well-defined; the proof that it is a UNTF is also left to the reader. To conclude, note that (iv) follows immediately from the definition of an (M,N)-PUNTF where $\lambda = \frac{M}{N}$. □

Note that the assumption $2N \leq M$ is crucial to the proof of Theorem 12; in the case where λ is slightly smaller than $\frac{M}{N}$, the (n_0+1)th diagonal entry of FF^* must accept nearly two spectral units of weight, which is only possible when the desired Spectral Tetris height $\frac{M}{N}$ is at least 2. At the same time, we note that playing Spectral Tetris can also result in matrices of lesser redundancy, provided larger blocks are used. Indeed, UNTFs of redundancy $\frac{M}{N} \geq \frac{3}{2}$ can be constructed using 3×3 Spectral Tetris submatrices, as we now have two diagonal entries over which to spread at most three units of spectral weight; the blocks themselves are obtained by scaling the rows of a 3×3 discrete Fourier transform matrix. More generally, UNTFs with redundancy greater than $\frac{J}{J-1}$ can be constructed using $J \times J$ submatrices. Note that these lower levels of redundancy are only bought at the expense of a loss in sparsity, and in particular, a loss of orthogonality relations between the frame elements themselves. We have focused on the use of 2×2 submatrices since, as we shall see in the next section, it is precisely these orthogonality relations which facilitate our modulated TFF construction. In particular, by playing Spectral Tetris with only 1×1 and 2×2 submatrices, that is, by repeatedly applying the rules of Theorem 12, one obtains a UNTF in which many frame elements are mutually orthogonal:

Theorem 13. *For any $M, N \in \mathbb{N}$ such that $2N \leq M$, there exists a unit norm tight frame $\{f_m\}_{m=1}^M$ for $\mathbb{C}^N$ with the property that $\langle f_m, f_{m'}\rangle = 0$ whenever $m' - m \geq \lfloor \frac{M}{N} \rfloor + 3$.*

Proof. Define $\{f_m^{(1)}\}_{m=1}^M$ in $\mathbb{C}^N$ by letting $f_1^{(1)} = e_1$ and setting the rest of the vectors to zero. One may quickly verify that this sequence is a $(1,1)$-PUNTF. Next, construct each successive $(m_0(j), n_0(j))$-PUNTF $\{f_m^{(j)}\}_{m=1}^M$ for $j = 1, \dots, J$ according to Theorem 12. Note that there is no uncertainty in this process; at each step, we must apply Theorem 12's rule (i), (ii) or (iii), resulting in an increase in m_0 by either 1 or 2. Take $J \leq M$ to be the last iteration, namely the first j such that $m_0(j) \geq M$. The contrapositives of (i), (ii) and (iii) imply $\lambda(J) = \frac{M}{N}$ and $n_0(J) = N$, and so by (iv), $\{f_m\}_{m=1}^M := \{f_m^{(J)}\}_{m=1}^M$ is indeed a UNTF $\{f_m^{(J)}\}_{m=1}^M$ for $\mathbb{C}^N$.

Now, fix $m = 1, \dots, M$, and suppose the j^* is the first index j for which $f_m^{(j)}$ is found in its final form, that is, $j^* := \min\{j : m_0(j) \geq m\}$. Considering (i), (ii) and (iii), we see that either $m_0(j^*) = m$ or $m_0(j^*) = m + 1$. In either case, $m_0(j^*) \leq m + 1$. Recall that after constructing f_m, we proceeded to construct $f_{m'}$ for $m' > m$ using repeated applications of (i), (ii) and (iii). If needed, we repeatedly applied (i), continually increasing $\lambda(j^*)$ by 1, until the new value was strictly greater than $\frac{M}{N} - 1$. Therefore, we applied (i) precisely $\lfloor \frac{M}{N} - \lambda(j^*) \rfloor$ times. At this point, we were either finished, by (iv), or continued our construction using (ii) or (iii); in either of these two latter cases, we thus increased $n_0(j^*)$ by 1, thereby ensuring that any newly constructed $f_{m'}$'s were orthogonal to f_m, having disjoint support. That is, $\langle f_m, f_{m'}\rangle = 0$ whenever m' is greater than the value obtained by first increasing $m_0(j^*)$ by $\lfloor \frac{M}{N} - \lambda(j^*) \rfloor$ and then further increasing $m_0(j^*)$ by either 1 or 2, using rules (iii) or (ii), respectively. In short, $\langle f_m, f_{m'}\rangle = 0$ whenever:

$$m_0(j^*) + \lfloor \tfrac{M}{N} - \lambda(j^*) \rfloor + 2 < m'. \tag{14}$$

Noting that the definition of a PUNTF gives $\lambda(j^*) \geq 1$ and recalling that $m_0(j^*) \leq m + 1$, the left hand side of (14) can be bounded above by:

$$m_0(j^*) + \lfloor \tfrac{M}{N} - \lambda(j^*) \rfloor + 2 \leq (m + 1) + \lfloor \tfrac{M}{N} - 1 \rfloor + 2 = m + \lfloor \tfrac{M}{N} \rfloor + 2.$$

Thus, in order to satisfy (14), it suffices to have $m + \lfloor \frac{M}{N} \rfloor + 2 < m'$, that is, $m' - m \geq \lfloor \frac{M}{N} \rfloor + 3$, as claimed. □

When $M = 11$, $N = 4$, the previous result states that the UNTF of Example 10 satisfies $\langle f_m, f_{m'}\rangle = 0$ whenever $m' - m \geq 5$. Moreover, since $\langle f_7, f_3\rangle \neq 0$, we see that Theorem 13's condition on $m' - m$ is, in fact, the best possible.

Also note that although this section's results were proved in complex Euclidean space for the sake of consistency, the frames obtained by playing Spectral Tetris with 1×1 and 2×2 submatrices are, in fact, real-valued. We believe the simplicity of this construction rivals that of real harmonic frames, consisting of samples of sines and cosines. In particular, Spectral Tetris provides a very simple proof of the existence of real UNTFs for any $M \geq N$: when $2N \leq M$, the construction is direct; Naimark complements then give real UNTFs with redundancy less than two.

4. Modulated fusion frames

In this section, we provide the second half of a general method for constructing (K, L, N)-TFFs when $K \geq \lceil \frac{N}{L} \rceil + 2$. The key idea is to modulate UNTFs whose frame elements satisfy certain orthogonality relations, such as those provided by Theorem 13:

Theorem 14. *If $\{f_n\}_{n=1}^N$ is a UNTF for $\mathbb{C}^L$ and $\langle f_n, f_{n'}\rangle = 0$ whenever K divides $n' - n \neq 0$, then $\{g_{k,l}\}_{k=1,\,l=1}^{K\ \ L} \subseteq \mathbb{C}^N$*

$$g_{k,l}(n) = \frac{\sqrt{L}}{\sqrt{N}}\, \mathrm{e}^{2\pi \mathrm{i}(k-1)n/K} f_n(l)$$

generates a (K, L, N)-TFF for $\mathbb{C}^N$.

Proof. We show that $\{g_{k,l}\}_{k=1,\,l=1}^{K\ \ L}$ satisfies Definition 3. In particular, for any $k = 1, \dots, K$, the fact that $\{f_n\}_{n=1}^N$ is a UNTF for $\mathbb{C}^L$ implies:

$$\langle g_{k,l}, g_{k,l'}\rangle = \tfrac{L}{N} \sum_{n=1}^{N} \mathrm{e}^{2\pi \mathrm{i}(k-1)n/K} f_n(l) \overline{\mathrm{e}^{2\pi \mathrm{i}(k-1)n/K} f_n(l')} = \tfrac{L}{N} \sum_{n=1}^{N} f_n(l) \overline{f_n(l')} = \begin{cases} 1, & l = l', \\ 0, & l \neq l', \end{cases}$$

as needed. Furthermore, (4) is also satisfied:

$$\begin{aligned}\sum_{k=1}^{K}\sum_{l=1}^{L} g_{k,l}(n)\overline{g_{k,l}(n')} &= \tfrac{L}{N}\sum_{k=1}^{K}\sum_{l=1}^{L} \mathrm{e}^{2\pi\mathrm{i}(k-1)n/K} f_n(l)\overline{\mathrm{e}^{2\pi\mathrm{i}(k-1)n'/K} f_{n'}(l)}\\ &= \tfrac{L}{N}\sum_{l=1}^{L} f_n(l)\overline{f_{n'}(l)}\sum_{k=0}^{K-1}\mathrm{e}^{2\pi\mathrm{i}k(n-n')/K}\\ &= \tfrac{L}{N}\langle f_n, f_{n'}\rangle \begin{cases} K, & K \mid n'-n,\\ 0, & K \nmid n'-n,\end{cases}\\ &= \begin{cases} \frac{KL}{N}, & n = n',\\ 0, & n \neq n',\end{cases}\end{aligned}$$

where the final equality follows from the assumption that $\langle f_n, f_{n'}\rangle = 0$ whenever K divides $n' - n \neq 0$. □

We note that the frame vectors produced by Theorem 14 are not modulates of the original frame vectors themselves, but rather their coordinate vectors. That is, the analysis operator of $\{g_{k,l}\}_{k=1,\,l=1}^{K\ \ \,L}$ is obtained by vertically stacking modulated copies of the synthesis operator of $\{f_n\}_{n=1}^{N}$, as illustrated in the following example.

Example 15. Recall that, by Theorem 13, the UNTF $\{f_m\}_{m=1}^{11}$ for $\mathbb{C}^4$ constructed in Example 10 satisfies $\langle f_m, f_{m'}\rangle = 0$ whenever $m' - m \geq 5$. Applying Theorem 14 to this UNTF with $K = 5$ produces a $(5, 4, 11)$-TFF, as given in Table 1.

We now apply this idea in general, using Theorem 14 to modulate the Spectral Tetris constructions of Theorem 13. As seen in Table 1, this results in a collection of vectors which, from afar, appear as translates and modulates of a single function. These *modulated fusion frames* provide the final ingredient for the proof of our first main result:

Proof of Theorem 1. Take any $K, L, N \in \mathbb{N}$ such that $L \leq N$. The case where L divides N is resolved in Corollary 5. Assuming $2L < N$, where L does not divide N, the necessary condition on the existence of (K, L, N)-TFFs is given by Corollary 9. For the sufficient condition, further assume that $K \geq \lceil \frac{N}{L} \rceil + 2$. Since $2L < N$, Theorem 13 provides a UNTF $\{f_n\}_{n=1}^{N}$ for $\mathbb{C}^L$ that has the property that $\langle f_n, f_{n'}\rangle = 0$ whenever $n' - n \geq \lfloor \frac{N}{L} \rfloor + 3 = \lceil \frac{N}{L} \rceil + 2$. For any $K \geq \lceil \frac{N}{L} \rceil + 2$, applying Theorem 14 to this UNTF produces a (K, L, N)-TFF. □

We note that the proof of Theorem 1 is entirely constructive, building TFFs either in terms of tensor products or as modulated fusion frames. However, this result is not a comprehensive characterization of existence. In the next section, we resolve any remaining ambiguity by proving our second main result, namely Theorem 2.

5. The Tight Fusion Frame Existence Test

In this section, we complete the characterization of the existence of equal-rank TFFs. In particular, we prove Theorem 2 by showing that for a given $K, L, N \in \mathbb{N}$ with $L < N$, the Tight Fusion Frame Existence Test (TFFET) given in Table 2 will terminate in at most L iterations of its "while" loop. In particular, TFFET resolves the question of existence of (K, L, N)-TFFs in the case where the triple (K, L, N) is ambiguous with respect to Theorem 1, that is, when (K, L, N) satisfies $2L < N$, L does not divide N, and $K = \lceil \frac{N}{L} \rceil + 1$. In short, we now show that no more than L successive applications of Naimark and spatial complements will inevitably relate an ambiguous triple to one that is not ambiguous:

Proof of Theorem 2. Pick $K, L, N \in \mathbb{N}$ such that $L < N$. As seen in Line 2 of TFFET, let $(K, L_0, N_0) = (K, L, N)$ if $2L \leq N$, and let $(K, L_0, N_0) = (K, N - L, N)$ otherwise. By invoking Corollary 8.i if necessary, we have that (K, L, N)-TFFs exist if and only if (K, L_0, N_0)-TFFs exist. For the $(j + 1)$st iteration of the loop that begins on Line 4, note that if L_j divides N_j, then the existence of (K, L_j, N_j)-TFFs is characterized by Theorem 1, as implemented in Lines 5–7 of TFFET. Moreover, Theorem 1 also characterizes existence whenever L_j does not divide N_j and $K \neq \lceil \frac{N_j}{L_j} \rceil + 1$, as seen in TFFET Lines 8–10. All that remains to be resolved is the ambiguous case where $2L_j < N_j$, L_j does not divide N_j, and $K = \lceil \frac{N_j}{L_j} \rceil + 1$.

Table 1: The analysis operator of a $(5,4,11)$-TFF, as described in Example 15. Here, $w := \mathrm{e}^{-2\pi\mathrm{i}/5}$. The rows of this matrix form a TFF for $\mathbb{C}^{11}$ consisting of 5 subspaces, each of dimension 4. Here, a given pair of rows belong to the same subspace if their indices differ by a multiple of 5.

$$\begin{bmatrix}
1 & 1 & \frac{\sqrt{3}}{\sqrt{8}} & \frac{\sqrt{3}}{\sqrt{8}} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & w & \frac{\sqrt{3}}{\sqrt{8}}w^2 & \frac{\sqrt{3}}{\sqrt{8}}w^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & w^2 & \frac{\sqrt{3}}{\sqrt{8}}w^4 & \frac{\sqrt{3}}{\sqrt{8}}w^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & w^3 & \frac{\sqrt{3}}{\sqrt{8}}w & \frac{\sqrt{3}}{\sqrt{8}}w^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & w^4 & \frac{\sqrt{3}}{\sqrt{8}}w^3 & \frac{\sqrt{3}}{\sqrt{8}}w & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{\sqrt{5}}{\sqrt{8}} & -\frac{\sqrt{5}}{\sqrt{8}} & 1 & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}} & 0 & 0 & 0 & 0\\
0 & 0 & \frac{\sqrt{5}}{\sqrt{8}}w^2 & -\frac{\sqrt{5}}{\sqrt{8}}w^3 & w^4 & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}}w & 0 & 0 & 0 & 0\\
0 & 0 & \frac{\sqrt{5}}{\sqrt{8}}w^4 & -\frac{\sqrt{5}}{\sqrt{8}}w & w^3 & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}}w^2 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{\sqrt{5}}{\sqrt{8}}w & -\frac{\sqrt{5}}{\sqrt{8}}w^4 & w^2 & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}}w^3 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{\sqrt{5}}{\sqrt{8}}w^3 & -\frac{\sqrt{5}}{\sqrt{8}}w^2 & w & \frac{\sqrt{2}}{\sqrt{8}} & \frac{\sqrt{2}}{\sqrt{8}}w^4 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}} & 1 & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & 0\\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}}w & w^2 & \frac{1}{\sqrt{8}}w^3 & \frac{1}{\sqrt{8}}w^4 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}}w^2 & w^4 & \frac{1}{\sqrt{8}}w & \frac{1}{\sqrt{8}}w^3 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}}w^3 & w & \frac{1}{\sqrt{8}}w^4 & \frac{1}{\sqrt{8}}w^2 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{\sqrt{8}} & -\frac{\sqrt{6}}{\sqrt{8}}w^4 & w^3 & \frac{1}{\sqrt{8}}w^2 & \frac{1}{\sqrt{8}}w & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{\sqrt{8}} & -\frac{\sqrt{7}}{\sqrt{8}} & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{\sqrt{8}}w^3 & -\frac{\sqrt{7}}{\sqrt{8}}w^4 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{\sqrt{8}}w & -\frac{\sqrt{7}}{\sqrt{8}}w^3 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{\sqrt{8}}w^4 & -\frac{\sqrt{7}}{\sqrt{8}}w^2 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{\sqrt{8}}w^2 & -\frac{\sqrt{7}}{\sqrt{8}}w & 1
\end{bmatrix}$$

Table 2: The Tight Fusion Frame Existence Test (TFFET). As shown in the proof of Theorem 2, applying this test to any given $K, L, N \in \mathbb{N}$, $L < N$, will resolve the existence of (K, L, N)-TFFs in no more than L iterations of its "while" loop.

```
set K,L,N ∈ ℕ, L < N
if 2L > N, L := N − L
exists := 'unknown'
while exists := 'unknown'
  if L | N
    if K ≥ N/L, exists := 'true'
    else exists := 'false'
  else
    if K > ⌈N/L⌉ + 1, exists := 'true'
    else if K < ⌈N/L⌉ + 1, exists := 'false'
    else N := KL − N, L := N − L
end while
```

In this case, we necessarily have $L_j < KL_j - N_j$, and so we can apply Corollary 8.ii and then Corollary 8.i to obtain that (K, L_j, N_j)-TFFs exist if and only if $(K, L_{j+1}, N_{j+1}) := (K, (K-1)L_j - N_j, KL_j - N_j)$-TFFs exist. In TFFET, the reduction of (K, L_j, N_j) to (K, L_{j+1}, N_{j+1}) is accomplished in Line 11. In essence, TFFET's "while" loop first checks whether Theorem 1 resolves the existence of (K, L_j, N_j)-TFFs; in the case where it does not, TFFET instead calculates the alternative triple (K, L_{j+1}, N_{j+1}) for which the question of TFF existence is equivalent to that of the original. Note that the full utility of Theorem 1 is predicated upon whether $2L < N$; it is therefore important to note that whenever a given triple (K, L_j, N_j) is ambiguous, we have $K = \lceil \frac{N_j}{L_j} \rceil + 1 < \frac{N_j}{L_j} + 2$, and so $2L < N$ also holds for the new triple:

$$2L_{j+1} = 2[(K-1)L_j - N_j] = KL_j + [(K-2)L_j - 2N_j)] < (\tfrac{N_j}{L_j} + 2)L_j + [(K-2)L_j - 2N_j)] = KL_j - N_j = N_{j+1}.$$

Thus, we see that TFFET, starting from a given (K, L, N), will produce a sequence of triples for which the question of TFF existence is equivalent to that of (K, L, N). To show that TFFET completely characterizes the existence of equal-rank TFFs, we therefore need only show that its "while" loop terminates after a finite number of steps. Indeed, we claim that for some $J = 0, \dots, L_0 - 1$, the existence of (K, L_J, N_J)-TFFs is resolved by Theorem 1. To see this, recall that $L_{j+1} = (K-1)L_j - N_j = \lceil \frac{N_j}{L_j} \rceil L_j - N_j$ where L_j does not divide N_j, and so $0 < L_{j+1} < L_j$. As such, L_j decreases by at least 1 at each iteration, and remains positive. Thus, TFFET terminates within L_0 iterations of its "while" loop: if it does not terminate before the L_0th step, the final iteration simply determines whether $(K, 1, N_{L_0-1})$-TFFs exist, by invoking Lines 5–7. □

We conclude this paper by using TFFET to find a closed form expression of all $K, L, N \in \mathbb{N}$ for which (K, L, N)-TFFs do not exist.

5.1. *Levels of ambiguity*

Take any $K, L, N \in \mathbb{N}$ where, in light of Corollary 8.i, we assume without loss of generality that $2L \leq N$. We define the *level of ambiguity* of (K, L, N) to be one less than the number of iterations of TFFET's "while" loop that is necessary to resolve the existence of corresponding TFFs. In particular, (K, L, N) is 1-ambiguous whenever it is ambiguous but the spatial complement of its Naimark complement is not ambiguous. Triples of higher ambiguity may be characterized by reversing TFFET's analysis, that is, by repeatedly taking the Naimark complements of the spatial complements of 1-ambiguous triples:

Theorem 16. *Take any $K, L, N \in \mathbb{N}$ such that $2L \leq N$. If (K, L, N) is ambiguous, then $K \geq 4$. Moreover, all J-ambiguous triples (K, L, N) for which (K, L, N)-TFFs do not exist are of the form (K, N_J, L_J),*

$$L_J = \begin{cases} \frac{\alpha^{J-1} - \beta^{J-1}}{\alpha - \beta} N_1 - \frac{(\alpha+1)\alpha^{J-2} - (\beta+1)\beta^{J-2}}{\alpha - \beta} L_1, & K > 4, \\ L_1 + (J-1)(N_1 - 2L_1), & K = 4, \end{cases} \tag{15}$$

$$N_J = \begin{cases} \frac{(\alpha+1)\alpha^{J-1} - (\beta+1)\beta^{J-1}}{\alpha - \beta} N_1 - \frac{(\alpha+1)^2\alpha^{J-2} - (\beta+1)^2\beta^{J-2}}{\alpha - \beta} L_1, & K > 4, \\ N_1 + 2(J-1)(N_1 - 2L_1), & K = 4, \end{cases} \tag{16}$$

where $K, L_1, N_1 \in \mathbb{N}$ are any numbers for which $K \geq 4$, L_1 does not divide N_1, $2L_1 < N_1$, and:

$$1 \leq (K-2)[(K-1)L_1 - N_1] < L_1, \qquad (K-1)[(K-1)L_1 - N_1] \neq L_1. \tag{17}$$

Here, $\alpha := \frac{1}{2}\Big(K - 2 + \sqrt{K^2 - 4K}\Big)$, $\beta := \frac{1}{2}\Big(K - 2 - \sqrt{K^2 - 4K}\Big)$.

Proof. Recall that if (K, L, N) is ambiguous, then $2L < N$, L does not divide N and $K = \lceil \frac{N}{L} \rceil + 1$. In particular, since $\frac{N}{L} > 2$, then $K \geq 4$. Having this fact, we next characterize all 1-ambiguous *blank* triples (ABT) (K, L_1, N_1), that is, 1-ambiguous triples for which a corresponding TFF does not exist.

Indeed, fixing $L_1, N_1 \in \mathbb{N}$ such that L_1 does not divide N_1 and $2L_1 < N_1$, note that (K, L_1, N_1) is ambiguous if and only if $K = \lceil \frac{N_1}{L_1} \rceil + 1$, that is, if and only if $0 < R < L$, where $R := (K-1)L_1 - N_1$. At the same time, taking Naimark and then spatial complements of (K, L_1, N_1) yields $(K, (K-1)L_1 - N_1, KL_1 - N_1) = (K, R, L_1 + R)$. As such, (K, L_1, N_1) is a 1-ABT if and only if $0 < R < L$ and either $K < \frac{L_1 + R}{R} = \frac{L_1}{R} + 1$ when R divides L_1 or $K < \lceil \frac{L_1 + R}{R} \rceil + 1 = \lceil \frac{L_1}{R} \rceil + 2$ when R does not divide L_1. Since $K \geq 4$ and R is an integer, we may reduce these three conditions to two: either

$1 \leq R < \frac{L_1}{K-1}$ when R divides L_1 or $1 \leq R < \frac{L_1}{K-2}$ when R does not. Moreover, a basic arithmetic argument shows if R divides L_1 and $\frac{L_1}{K-1} \leq R < \frac{L_1}{K-2}$, then $K-1$ necessarily divides L_1 and $R = \frac{L_1}{K-1}$. Thus, we see that (K, L_1, N_1) is a 1-ABT if and only if $1 \leq R < \frac{L_1}{K-2}$ and $R \neq \frac{L_1}{K-1}$, namely (17).

We now use this characterization of 1-ABTs to find all ABTs. Indeed, recalling TFFET, the spatial complement of the Naimark complement of a j-ABT is a $(j-1)$-ABT. Reversing this process, we see that every J-ABT may be obtained by taking $J-1$ Naimark-of-spatial complements of a 1-ABT. We therefore can use induction to verify (15) and (16) for all $J \geq 1$. Indeed (15) and (16) are tautologies when $J = 1$, as they state $L_1 = L_1$ and $N_1 = N_1$, respectively; the only conditions on L_1 and N_1 are those given in (17). To elaborate, (15) and (16) obviously hold when $J = 1$ and $K = 4$. Moreover, since $\alpha\beta = 1$, the coefficients of L_1 in (15) and (16) have the following numerators, respectively:

$$(\alpha+1)\alpha^{1-2} - (\beta+1)\beta^{1-2} = \tfrac{\beta-\alpha}{\alpha\beta} = -(\alpha-\beta), \qquad (\alpha+1)^2\alpha^{1-2} - (\beta+1)^2\beta^{1-2} = (\alpha-\beta)(1-\tfrac{1}{\alpha\beta}) = 0.$$

Thus (15) and (16) also hold when $J = 1$ and $K > 4$.

Now assume that (15) and (16) hold for a given J. Taking Naimark-of-spatial complements of (K, L_J, N_J) produces $(K, L_{J+1}, N_{J+1}) = (K, N_J - L_J, K(N_J - L_J) - N_J)$. When $K = 4$, one may quickly verify that L_{J+1} and N_{J+1} are indeed given by (15) and (16), respectively. Next, for $K > 4$, a straightforward computation reveals that $L_{J+1} = N_J - L_J$ satisfies (15). Finally, since α and β are the solutions of the quadratic equation $(K-1)\gamma - 1 = \gamma(\gamma+1)$, we have that $N_{J+1} = KL_{J+1} - N_J$ is:

$$\begin{aligned} N_{J+1} &= K\tfrac{\alpha^J-\beta^J}{\alpha-\beta}N_1 - K\tfrac{(\alpha+1)\alpha^{J-1}-(\beta+1)\beta^{J-1}}{\alpha-\beta}L_1 - \tfrac{(\alpha+1)\alpha^{J-1}-(\beta+1)\beta^{J-1}}{\alpha-\beta}N_1 + \tfrac{(\alpha+1)^2\alpha^{J-2}-(\beta+1)^2\beta^{J-2}}{\alpha-\beta}L_1 \\ &= \tfrac{\alpha^{J-1}[(K-1)\alpha-1]-\beta^{J-1}[(K-1)\beta-1]}{\alpha-\beta}N_1 - \tfrac{(\alpha+1)\alpha^{J-2}[(K-1)\alpha-1]-(\beta+1)\beta^{J-2}[(K-1)\beta-1]}{\alpha-\beta}L_1 \\ &= \tfrac{(\alpha+1)\alpha^J-(\beta+1)\beta^J}{\alpha-\beta}N_1 - \tfrac{(\alpha+1)^2\alpha^{J-1}-(\beta+1)^2\beta^{J-1}}{\alpha-\beta}L_1, \end{aligned}$$

as claimed in (16). □

We conclude with an example of TFFET and the characterization provided by Theorem 16, noting that in the special case of $K = 4$, even small-valued triples can have high levels of ambiguity. In particular, $(4, 25, 53)$ has 8-ambiguity, meaning TFFET's "while" loop runs for 9 iterations:

$$(4,25,53) \to (4,22,47) \to (4,19,41) \to (4,16,35) \to (4,13,29) \to (4,10,23) \to (4,7,17) \to (4,4,11) \to (4,1,5).$$

If, on the other hand, $K > 4$, the entries of ambiguous blank triples grow geometrically in terms of the ambiguity.

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