

Erasure-Proof Transmissions: Fusion Frames meet Coding Theory

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ABSTRACT

In this paper we analyze the use of frames for the transmission and error-correction of analog signals via a memoryless erasure-channel. We measure performance in terms of the mean-square error remaining after error correction and reconstruction. Our results continue earlier works on frames as codes which were mostly concerned with the smallest number of erased coefficients. To extend these works we borrow some ideas from binary coding theory and realize them with a novel class of frames, which carry a particular fusion frame architecture. We show that a family of frames from this class achieves a mean-square reconstruction error remaining after corrections which decays faster than any inverse power in the number of frame coefficients.

Keywords: Bernoulli trials, Erasure channel, Fusion frames, Frames, Iterative Correction Scheme, Mean Square Error, Nested Frames, Reconstruction Error

1. INTRODUCTION

Coding theory has been extensively developed and applied with great success to the suppression of errors that occur when *digital* data is transmitted through an unreliable analog channel. On the other hand, today we often transmit *analog* data (images, audio or video signals) through a digital channel such as the internet which can be unreliable as well, due to data loss or undesirable delays. The theory of frames, serving for the encoding of analog signals in a redundant way, has proven useful for suppressing such types of transmission errors. Both worlds, the binary world of coding theory and the analog world of frame theory developed numerous tools for providing error-suppression.

However, in contrast to the information-theoretic approach to codes, a large number of works on frames and erasures have so far focused on optimal encoding for the smallest number of erasures. We continue this development by considering the linear transmission of vectors through a memoryless analog channel that randomly either transmits a coefficient perfectly or discards it — a commonly used, realistic model in coding theory. However, unlike traditional codes over finite fields, here we focus on ‘analog’ vectors, i.e., in a real or complex Hilbert space \mathcal{H} . We design a novel class of frames, which bear a particular fusion frame structure, specifically designed for this problem, and analyze its performance. This class is inspired by the work on error-free coding from the information theorist Peter Elias.⁸

Summarizing, for constructing erasure-proof encoding of analog signals and designing correction schemes, the interplay of both areas is an essential tool, which is a leitmotiv throughout the whole manuscript.

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1.1. Frames: Redundancy in the Analog World

Frame theory is the key to construct erasure-resilient coding for analog signals, hence accounting for the typically analog nature of signals. For an introduction which also encompasses various application areas, we refer the interested reader to the survey papers by Kovačević and Chebira.^{14, 15}

Let us first briefly review the basic definitions and notations related to frames. Loosely speaking, a frame is a generalization of orthonormal bases providing stable, non-unique (redundant) expansions with continuous coefficients. More precisely, a family of vectors $\mathcal{F} = \{f_j\}_{j \in J}$ in \mathcal{H} is called a *frame* if there exist constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{j \in J} |\langle x, f_j \rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathcal{H}.$$

If A and B can be chosen as $A = B$, then the frame is called *A-tight*. In case $A = B = 1$ is possible, \mathcal{F} is a *Parseval frame*. A frame is called *equal-norm*, if there exists some $c > 0$ such that all frame vectors satisfy $\|f_j\| = c$.

Apart from providing redundant expansions, frames can also serve as an analysis tool. In fact, they allow the analysis of data by studying the associated coefficients $(\langle x, f_j \rangle)_{j \in J}$, wherefore the map $V : \mathcal{H} \rightarrow \ell^2(J)$ defined by

$$V : \mathcal{H} \rightarrow \ell^2(J), \quad x \mapsto (\langle x, f_j \rangle)_{j \in J}$$

is termed the *analysis operator*. If \mathcal{F} is a Parseval frame, then we have perfect reconstruction from those coefficients by applying the adjoint operator, i.e.,

$$V^*V = I.$$

Sometimes it is advantageous, when modeling applications which require distributed processing or – such as in our situation – an iterative, hierarchically structured (error-correcting) procedure to use a redundant collection of subspaces. For this very purpose, fusion frames were introduced and studied by Casazza and Kutyniok et al.^{6, 7, 17} Fusion frames – which can from a different view point be regarded as a generalization of frames – have also been referred to as frames for subspaces in an earlier paper by Casazza and Kutyniok⁵ or weighted projections resolving the identity by Bodmann.²

A *fusion frame* is a set of subspaces $\{\mathcal{W}_i\}_{i \in I}$ in \mathcal{H} with associated weights $v_i > 0$ which satisfies

$$A\|x\|^2 \leq \sum_{i \in I} v_i^2 \|P_i x\|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathcal{H},$$

where P_i are the orthogonal projections onto the subspaces \mathcal{W}_i .

Taking a different view point, let $\mathcal{F} = \{f_j\}_{j \in J}$ be a frame in \mathcal{H} , which can be partitioned into subsets $\mathcal{F}_i = \{f_j\}_{j \in J_i}$, $i \in I$, such that J is the disjoint union of the sets J_i . If the subsets

$$\mathcal{W}_i = \text{span}\{f_j : j \in J_i\}, \quad i \in I,$$

form a fusion frame in the above sense, then also the frame \mathcal{F} itself together with the partitions \mathcal{F}_i might be called a *fusion frame*.

1.2. Erasure-Channels

Our model for an erasure-channel is a memoryless channel that either transmits a coefficient perfectly or discards it, in accordance with the outcomes of independent, identically distributed Bernoulli trials.

A communication system will be given by a frame \mathcal{F} for a Hilbert space \mathcal{H} . Now assume that we intend to transmit coefficients $(\langle x, f_j \rangle)_{j \in J}$ of a signal $x \in \mathcal{H}$, i.e., we transmit $Vx \in \ell^2(J)$. To model the occurring erasures, we let $\{\beta_j\}_{j \in J}$ be a family of independent identically distributed Bernoulli experiments with erasure probability $q = \mathbb{P}[\beta_j = 0]$, where $\beta_j = 1$ indicates that the j th coefficient is erased and $\beta_j = 0$ indicates that no

erasure happened. Further, let Δ be the random diagonal matrix with entries $\Delta_{j,j} = \beta_j$. The received vector is then of the form

$$(I - \Delta)Vx \in \ell^2(J). \quad (1)$$

We will later consider the transmission through the memoryless erasure-channel described above, followed by an error correction enabled by a particular choice of the frame. The error correction employed in this paper will be a combination of both, active error correction and blind reconstruction. The reconstruction error is then analyzed *after* error-correction by considering the mean-square error. The reader should be aware that this error-correction will need to be encoded into the family of binary ($\{0, 1\}$ -valued) random variables $\{\beta_j\}_{j \in J}$. We will detail this in Subsection 2.2.

1.3. Redundancy: Frames versus Codes

Redundancy in both frame theory and coding theory is used to allow for the possibility of error correction. However, there is a major difference in how redundancy is manifested in both theories:

- **FRAME THEORY.** We have ‘*soft redundancy*’ in the following sense: Once the number of erasures exceeds a particular value and perfect reconstruction is not possible anymore, the reconstruction error increases gradually.
- **CODING THEORY.** We have ‘*hard redundancy*’ in the following sense: Once the number of erasures exceeds a particular value and perfect reconstruction is not possible anymore, the reconstruction error typically cannot be controlled.

In order to provide a notion of redundancy which makes both worlds comparable, we define a coding rate for a given frame in analogy with binary codes. One of the goals in this paper will then be to show that for a fixed coding rate there are error correction algorithms which suppress the reconstruction error below any desired level.

DEFINITION 1.1. *Let \mathcal{H} be a Hilbert space of dimension d and \mathcal{F} a frame for \mathcal{H} consisting of n vectors. We say that \mathcal{F} has a coding rate of*

$$R = \frac{d}{n}.$$

Notice that, typically, the *inverse* of R is called the redundancy in the situation of an equal-norm tight frame.

1.4. Frame Theory and Erasures

Over the last years, the problem of reconstructing a vector in a finite-dimensional real or complex Hilbert space when not all of its frame coefficients are known has received much attention in the literature (see Refs. 4, 10–12, 20, 21). However, most of the results in the mathematically oriented literature focus on optimal performance for the smallest possible number of erased coefficients,^{3, 4, 12} which is not typical for transmissions via a memoryless erasure channel. Other results on so-called maximally robust frames guarantee recovery from a certain fraction of lost frame coefficients,¹⁹ but this may involve inverting an arbitrarily ill-conditioned matrix.

In this paper, we address the following problem: Given a fixed, sufficiently small probability of erased coefficients, find frames such that their associated coding rate remains bounded away from zero and such that the mean-square error remaining (after error correction is applied) decays rapidly in terms of the number of transmitted frame coefficients.

2. FRAME BASED CODING AND ERROR-CORRECTION SCHEME

We first discuss the novel coding scheme based on frame theoretic considerations we employ and the associated error-correction scheme.

2.1. Coding with product frames

The coding method relies on frames arising from tensor product constructions, which are a special type of a fusion frame.

DEFINITION 2.1. *Given Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_m$ and tight frames $\mathcal{F}^{(i)} = \{f_j^{(i)}\}_{j \in J_i}$ for each \mathcal{H}_i , then the family of vectors $\mathcal{F} = \{f_{j_1}^{(1)} \otimes f_{j_2}^{(2)} \otimes \dots \otimes f_{j_m}^{(m)} : j_i \in J_i \text{ for all } i\}$ is a tight frame for $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_m$. We call this frame \mathcal{F} a tight product frame.*

We note that if we fix all but one index, say the last, then the resulting set $f_{j_1}^{(1)} \otimes f_{j_2}^{(2)} \otimes \dots \otimes f_{j_{m-1}}^{(m-1)} \otimes \mathcal{F}^{(m)}$ is a tight frame for its span. Therefore, \mathcal{F} has a natural fusion frame architecture.

Similarly, fixing only the first $m - k$ indices of the frame vectors in the tensor product would provide a tight frame for a subspace for any $0 \leq k < m$. Moreover, there is a partial ordering on these tight fusion frames induced by the partial ordering of the subspaces they span.

2.2. Error-Correction Algorithm

After passing the frame coefficients through the erasure channel, an error-correction algorithm is applied. This error-correction is iterative and based on the hierarchical structure of the product frames introduced in the previous subsection.

We will see that in fact the error-correction scheme can be regarded as a combination of the two most widely used approaches, namely active error correction and blind reconstruction. When actively correcting erasures, one tries to fill in the values for the erased coefficients, and aims for a high probability of successfully restoring all lost coefficients. When blind reconstruction is used, one sets the missing coefficients to zero and reconstructs always in the same way. In this case, the usual goal is obtaining a small error norm. Loosely speaking, here we will correct coefficients as far as possible, and set the uncorrected remaining ones to zero.

To describe precisely the error-correction scheme we propose, let $\mathcal{F} = \mathcal{F}^{(1)} \otimes \dots \otimes \mathcal{F}^{(m)}$ be a product frame for $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_m$, and assume – as we will also do in the sequel – that each factor $\mathcal{F}^{(i)}$ can correct up to two erased frame coefficients. Then the error-correction scheme works as follows:

(1) For all $j_2 \in J_2, \dots, j_m \in J_m$:

(I) If at most two coefficients of the frame $F^{(1)} \otimes f_{j_2}^{(2)} \otimes \dots \otimes f_{j_m}^{(m)}$ are erased, then correct them.

(II) If more than two coefficients of the frame $F^{(1)} \otimes f_{j_2}^{(2)} \otimes \dots \otimes f_{j_m}^{(m)}$ are erased, then do nothing.

(2) For all $j_1 \in J_1$ and $j_3 \in J_3, \dots, j_m \in J_m$:

(I) If at most two coefficients of the frame $f_{j_1}^{(1)} \otimes F^{(2)} \otimes f_{j_3}^{(3)} \otimes \dots \otimes f_{j_m}^{(m)}$ are erased, then correct them.

(II) If more than two coefficients of the frame $f_{j_1}^{(1)} \otimes F^{(2)} \otimes f_{j_3}^{(3)} \otimes \dots \otimes f_{j_m}^{(m)}$ are erased, then do nothing.

(3) – $(m - 1)$ Continue this procedure iteratively.

(m) For all $j_1 \in J_1, j_2 \in J_2, \dots, j_{m-1} \in J_{m-1}$:

(I) If at most two coefficients of the frame $f_{j_1}^{(1)} \otimes f_{j_2}^{(2)} \otimes \dots \otimes f_{j_{m-1}}^{(m-1)} \otimes F^{(m)}$ are erased, then correct them.

(II) If more than two coefficients of the frame $f_{j_1}^{(1)} \otimes f_{j_2}^{(2)} \otimes \dots \otimes f_{j_{m-1}}^{(m-1)} \otimes F^{(m)}$ are erased, then do nothing.

The performance of this scheme is illustrated in Figure 1 for the case $m = 2$.

Let us mention that, in the general case, step (I) is performed if the erasures are *correctible*. We now elaborate on this. Given a Parseval frame $\mathcal{F} = \{f_1, f_2, \dots, f_n\}$ for \mathcal{H} with analysis operator V . Further let E be the diagonal $n \times n$ matrix with binary entries on the main diagonal, a 1 indicating that this coefficient is erased. If

$$(I - E)Vx$$

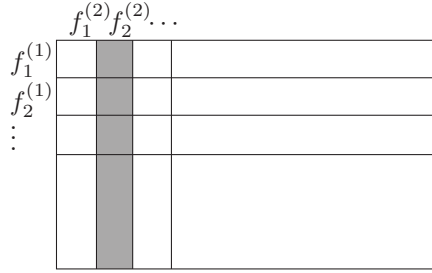


Figure 1. Error-Correction Scheme for $m = 2$. First the columns (here the column associated with $F^{(1)} \otimes f_2^{(2)}$ is marked) are corrected, followed by correction of the rows.

for some $x \in \mathcal{H}$ is the received vector (compare (1)), we can correct this vector if the positive operator $V^*(I - E)V$ has a bounded inverse. It is well-known that in this case, the inverse can be obtained from the norm-convergent Neumann series

$$(V^*(I - E)V)^{-1} = \sum_{n=0}^{\infty} (V^*EV)^n.$$

Applying this operator to the output of blind reconstruction $(I - E)Vx$ gives perfect reconstruction of the input vector x .

3. MEASURES FOR THE RECONSTRUCTION ERROR

We now need to introduce measures for the performance of the following scheme: Coding a signal $x \in \mathcal{H}$, say, by the coding scheme from Subsection 2.1, then passing it through the memoryless erasure channel described in Subsection 1.2 followed by the error-correction scheme from Subsection 2.2. As mentioned before our error-correction scheme can be regarded as a combination of active error correction and blind reconstruction.

Now our goal is to define a measure for average reconstruction performance when probabilities for erasures are known. To this end, we average the square of the reconstruction error with the distribution of erasures and input vectors. Here and hereafter, we denote the expectation of any random variable η with respect to the underlying probability measure \mathbb{P} by $\mathbb{E}[\eta] = \int \eta d\mathbb{P}$.

For simplicity, we have chosen a uniform measure on the unit sphere in \mathcal{H} for the input vectors and we assume that the distribution of input vectors is independent of that of the erasures. This performance measure seems to be motivated only by the blind reconstruction method, but we will show that we can deduce probabilities for correctibility from it.

DEFINITION 3.1. Let $\{\beta_j\}_{j \in J}$ be a family of binary ($\{0, 1\}$ -valued) random variables governed by a probability measure \mathbb{P} , and let Δ be the random diagonal matrix with entries $\Delta_{j,j} = \beta_j$. Moreover, let ξ be a random variable with values in the unit sphere $\{x \in \mathcal{H} : \|x\| = 1\}$ which is independent of the family $\{\beta_j\}$, and assume that the distribution of $U\xi$ is identical to that of ξ for any fixed unitary U . Given a Parseval frame \mathcal{F} for a Hilbert space \mathcal{H} with analysis operator V , we define the mean-square error by

$$\sigma^2(V, \beta) = \mathbb{E}[\|V^* \Delta V \xi\|^2].$$

There is a simple expression for the mean square error as the square of a weighted Frobenius norm of the Gramian VV^* .

LEMMA 3.2. Let $\{\beta_j\}_{j \in J}$ be as above, assume the family is identically distributed with probability $\mathbb{P}(\beta_j = 1) = q$, and assume the joint distribution is such that $\mathbb{P}(\beta_j = \beta_{j'} = 1) = r$ for all $j \neq j'$. Let Δ be the random diagonal

matrix with entries $\Delta_{j,j} = \beta_j$. If V is the analysis operator of a Parseval frame $\mathcal{F} = \{f_j\}_{j \in J}$ containing $n = |J|$ vectors in a Hilbert space \mathcal{H} of dimension d , then

$$\sigma^2(V, \beta) = \frac{1}{d} \left((q-r) \sum_{j=1}^n \|f_j\|^4 + r \sum_{j,l=1}^n |\langle f_j, f_l \rangle|^2 \right).$$

Proof. We note that for any fixed orthogonal projection D and vector $x \in \mathcal{H}$,

$$\|V^*DVx\|^2 = \langle (V^*DV)^2x, x \rangle.$$

Now integrating over ξ gives

$$\mathbb{E}[\|V^*DV\xi\|^2] = \frac{1}{d} \operatorname{tr}[(V^*DV)^2] = \frac{1}{d} \operatorname{tr}[(D V V^* D)^2].$$

In the last identity, we have used the cyclicity of the trace and the fact that D is a projection. Finally, replacing D by the random diagonal projection Δ and taking the expectation produces the claimed weights q on the diagonal and r for the off-diagonal entries in the squared Frobenius norm of the Grammian $V V^*$. \square

4. ERROR BOUNDS

4.1. General Analysis

We will now analyze how the previously introduced class of frames can be used to trade an increase in block length of encoding for better error correction capabilities.

We first consider the simplest case in which \mathcal{H} has two factors, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Also, as preparation for our main theorem, we first consider packet erasures (see Bodmann²) instead of erasures for single frame coefficients. This means, we have a frame $\mathcal{F} = \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}$ and a two-parameter family of random variables $\{\beta_{j,j'}\}$ which govern erasures of frame coefficients in such a way that either all coefficients belonging to some j' are erased or all of them are left intact. We now compute the mean-square error for this error model.

PROPOSITION 4.1. *Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and let V_1 and V_2 be the analysis operators of Parseval frames $\mathcal{F}^{(1)} = \{f_j^{(1)}\}_{j \in J_1}$ and $\mathcal{F}^{(2)} = \{f_{j'}^{(2)}\}_{j' \in J_2}$ for \mathcal{H}_1 and \mathcal{H}_2 having dimension d_1 and d_2 , respectively. Let $\{\beta_{j,j'} : j \in J_1, j' \in J_2\}$ be a two-parameter family of binary random variables which have probabilities $\mathbb{P}(\beta_{j,j'} = 1) = q$ and are distributed such that there is a family $\{\beta_{j'}^{(2)}\}_{j' \in J_2}$ and $\beta_{j,j'} = \beta_{j'}^{(2)}$ almost surely, regardless of j . The mean-square error for the frame \mathcal{F} and this type of packet erasures reduces to that of $\mathcal{F}^{(2)}$, that is*

$$\sigma^2(V_1 \otimes V_2, \beta) = \sigma^2(V_2, \beta^{(2)}).$$

Proof. By the assumption, the random erasure matrices associated with $\{\beta_{j,j'}\}$ are of the form $\Delta = I \otimes \Delta_2$, where the diagonal of Δ_2 is given by $\beta^{(2)}$. For any orthogonal projection matrix of the form $I \otimes D$, the squared Frobenius norm of the compressed Grammian satisfies

$$\operatorname{tr}[(I \otimes D)V_1V_1^* \otimes V_2V_2^*(I \otimes D)]^2 = \operatorname{tr}[(V_1V_1^*)^2] \operatorname{tr}[(DV_2V_2^*D)^2].$$

Inserting this expression in the computation of the mean-squared error gives

$$\sigma^2(V_1 \otimes V_2, \beta) = \frac{1}{d_1 d_2} \operatorname{tr}[(V_1V_1^*)^2] \mathbb{E}[\operatorname{tr}[(\Delta_2 V_2 V_2^* \Delta_2)^2]] = \frac{1}{d_2} \mathbb{E}[\operatorname{tr}[(\Delta_2 V_2 V_2^* \Delta_2)^2]].$$

In the last step, we have used that $V_1V_1^*$ is an orthogonal projection of rank d_1 . Now the usual expression for the mean square error related to V_2 and $\beta^{(2)}$ has emerged. \square

Next, we continue with three combinatorial lemmata. They prepare the main result which concerns the error correction capabilities of tight product frames. The main problem we wish to address with this result is the

following: Given a fixed, sufficiently small erasure probability q , find frames such that their associated coding rate is bounded away from zero and the mean-square error remaining after error correction is applied decays fast in terms of the number of frame vectors.

We show hereafter that product frames of the form $\mathcal{F} = \mathcal{F}^{(1)} \otimes \dots \otimes \mathcal{F}^{(m)}$, for which each factor $\mathcal{F}^{(i)}$ can correct up to two erased frame coefficients, satisfy the desired properties.

LEMMA 4.2. *Let $n_1 \geq 3$ and let $\{\beta_1, \beta_2, \dots, \beta_{n_1}\}$ be a family of independent, identically distributed random variables which take values in $\{0, 1\}$. Let $q_0 = \mathbb{P}(\beta_1 = 1)$ and $q_1 = \mathbb{P}(\sum_{j=1}^{n_1} \beta_j \geq 3)$, then*

$$q_1 \leq \frac{1}{6} n_1^3 q_0^3.$$

Proof. We compute

$$q_1 = 1 - (1 - q_0)^{n_1} - n_1(1 - q_0)^{n_1-1}q_0 - \frac{1}{2}n_1(n_1 - 1)(1 - q_0)^{n_1-2}q_0^2,$$

and by taking the derivative with respect to q_0 obtain

$$q_1' = \frac{1}{2}n_1(n_1 - 1)(n_1 - 2)(1 - q_0)^{n_1-3}q_0^2 \leq \frac{1}{2}n_1^3q_0^2.$$

Integrating again gives

$$q_1 \leq \frac{1}{6}n_1^3q_0^3,$$

which was our claim. \square

The probability estimated in this lemma is that of a packet of n_1 coefficients remaining corrupted after an error correction protocol has been applied which can correct any two erased coefficients.

By iteration, we obtain a simple consequence.

LEMMA 4.3. *Let $\{n_i\}_{i=1}^m$ be the sizes of index sets $\{J_i\}_{i=1}^m$, with $n_i \geq 3$ for all $i \in \{1, 2, \dots, m\}$. Assume there is an m -parameter family of binary, independent identically distributed random variables $\{\beta_{j_1, j_2, \dots, j_m}\}$ and associated families $\{\beta'_{j_2, j_3, \dots, j_m}\}$, $\{\beta''_{j_3, j_4, \dots, j_m}\}$, \dots , $\{\beta_{j_m}^{(m-1)}\}$ which are iteratively defined by $\beta_{j_1, j_2, \dots, j_m}^{(0)} \equiv \beta_{j_1, j_2, \dots, j_m}$ and*

$$\beta_{j_{k+1}, j_{k+2}, \dots, j_m}^{(k)} = \begin{cases} 1, & \text{if } \sum_{j_k=1}^{n_k} \beta_{j_k, j_{k+1}, \dots, j_m}^{(k-1)} \geq 3, \\ 0, & \text{else.} \end{cases}$$

If $\mathbb{P}(\beta_{1,1,\dots,1} = 1) = q_0$, then the family $\{\beta_j^{(m-1)}\}$ is independent, identically distributed with

$$\mathbb{P}(\beta_j^{(m-1)} = 1) \leq 6^{-\frac{1}{2}(3^{m-1}-1)} n_{m-1}^{3^1} n_{m-2}^{3^2} \dots n_1^{3^{m-1}} q_0^{3^{m-1}}.$$

Proof. At each iteration step, subsets of independent random variables are used to define the random variables at the next level, which are therefore also independent. The probability follows from simple power counting under the iteration. \square

The probability computed in the above lemma is the probability of a corrupted block remaining after applying erasure correction iteratively. The next lemma considers what happens when the error correction is applied to packets at the final level. Here, we deviate from the strategy of only reconstructing nontrivially when at most two packets are missing. Instead, we correct for missing packets and compute the probabilities for the residual mean-square error.

LEMMA 4.4. Let $\{\beta_1, \beta_2, \dots, \beta_n\}$, $n \geq 1$, be independent, identically distributed binary random variables with probability $\mathbb{P}(\beta_1 = 1) = q$. Let the random variables $\gamma_1, \gamma_2, \dots, \gamma_n$ be defined by $\gamma_j = \beta_j$ if $\sum_{j=1}^n \beta_j \geq 3$, and otherwise $\gamma_j = 0$ for all $j \in \{1, 2, \dots, n\}$. Then, for any j ,

$$\mathbb{P}(\gamma_j = 1) \leq \frac{1}{2}n^2q^3,$$

and for $j_1 \neq j_2$, we have

$$\mathbb{P}(\gamma_{j_1} = \gamma_{j_2} = 1) \leq nq^3.$$

Proof. For the first estimate we use that the probability of the event $\{\gamma_j = 1\}$ is the same if we intersect it with $\{\sum_{j=1}^n \beta_j \geq 3\}$. This means that, in addition to β_j , there must be at least two other random variables with value one. However, then we can replace each γ_j by β_j , and these are independent, hence

$$\mathbb{P}(\gamma_j = 1) = \mathbb{P}(\beta_j = 1 \text{ and } \sum_{j=1}^n \beta_j \geq 3) = q \left(1 - (1-q)^{n-1} - (n-1)(1-q)^{n-2}q \right).$$

Taking the derivative of the quantity in parentheses, estimating, and integrating again in the same way as in the proof of Lemma 4.2 gives

$$\mathbb{P}(\gamma_j = 1) \leq \frac{1}{2}n^2q^3.$$

The family $\{\gamma_j\}$ is not independent, but we can bound the probability $\mathbb{P}(\gamma_{j_1} = \gamma_{j_2} = 1)$ in an analogous way by intersecting the event with $\{\sum_{j=1}^n \beta_j \geq 3\}$. This yields

$$\mathbb{P}(\gamma_{j_1} = \gamma_{j_2} = 1) = q^2(1 - (1-q)^{n-2}) \leq nq^3.$$

The lemma is proved. \square

These lemmata allow us to formulate an error bound for the remaining mean-square error for blind reconstruction after our error correction protocol has been applied.

THEOREM 4.5. Let $V = V_1 \otimes V_2 \otimes \dots \otimes V_m$ be the analysis operator of a Parseval product frame $\mathcal{F} = \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)} \otimes \dots \otimes \mathcal{F}^{(m)}$ for a Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_m$. Denote the dimension of each \mathcal{H}_i by d_i and the number of frame vectors in $\mathcal{F}^{(i)}$ by n_i . Let $\{\beta_{j_1, j_2, \dots, j_m}\}$ be an m -parameter family of binary independent, identically distributed random variables, define $\{\beta_j^{(m-1)}\}$ as above, and let

$$\gamma_{j_1, j_2, \dots, j_m} = \begin{cases} \beta_{j_m}^{(m-1)}, & \text{if } \sum_{j_m=1}^{n_m} \beta_{j_m}^{(m-1)} \geq 3, \\ 0, & \text{otherwise,} \end{cases}$$

then

$$\sigma^2(V, \gamma) \leq \frac{1}{d_m} \left((q_m - r_m) \sum_{j=1}^{n_m} \|f_j^{(m)}\|^4 + r_m \sum_{j,l=1}^{n_m} |\langle f_j^{(m)}, f_l^{(m)} \rangle|^2 \right)$$

with

$$q_m = \frac{1}{2} 6^{-3/2(3^{m-1}-1)} n_m^2 n_{m-1}^3 n_{m-2}^3 \dots n_1^3 q_0^3$$

and

$$r_m = \frac{2}{n_m} q_m.$$

Proof. The claim follows immediately by combining the preceding lemmata with the expression for the mean-square error for packet erasures. \square

COROLLARY 4.6. *If $V = V_1 \otimes V_2 \otimes \dots \otimes V_m$ and all V_i belong to equal-norm Parseval frames, then it is well known that $\|f_j^{(i)}\|^2 = \frac{d_i}{n_i}$ and $|\langle f_j^{(i)}, f_l^{(i)} \rangle|^2 \leq d_i^2/n_i^2$. Thus, we have*

$$\sigma^2(V, \gamma) \leq q_m \frac{d_m}{n_m} + r_m d_m = 3q_m \frac{d_m}{n_m}$$

with

$$q_m = \frac{1}{2} 6^{-3/2(3^{m-1}-1)} n_m^2 n_{m-1}^3 n_{m-2}^3 \dots n_1^3 q_0^{3^m}.$$

4.2. An Example

As an example we consider the product frame $\mathcal{F} = \mathcal{F}^{(1)} \otimes \dots \otimes \mathcal{F}^{(m)}$ satisfying that $\mathcal{F}^{(i)}$ consists of $n_i = i^2 n_1$ vectors for each $i \in \{1, 2, \dots, m\}$ and $n_1 \geq 3$. Let the dimension of the Hilbert space \mathcal{H}_i spanned by $\mathcal{F}^{(i)}$ be

$$\dim(\mathcal{H}_i) = i^2 n_1 - 2,$$

and assume the frame can correct any two erased coefficients. Examples of such frames, which are even of equal-norm, are the harmonic ones.

The tensor product of these m Hilbert spaces has dimension

$$\dim\left(\bigotimes_{i=1}^m \mathcal{H}_i\right) = (m!)^2 n_1^m \prod_{i=1}^m \left(1 - \frac{2}{i^2 n_1}\right).$$

This means, the coding rate R of \mathcal{F} is bounded, independently of m , by

$$\begin{aligned} R &> \prod_{i=1}^m \left(1 - \frac{2}{i^2 n_1}\right) > \left(1 - \frac{2}{n_1}\right) \left(1 - \frac{2}{n_1} \sum_{i=2}^{\infty} \frac{1}{i^2}\right) \\ &= \left(1 - \frac{2}{n_1}\right) \left(1 - \frac{2}{6n_1} \left(\frac{\pi^2}{6} - 1\right)\right). \end{aligned}$$

It is straightforward to check that $n_1 \geq 3$ ensures $R > 0$.

The preceding theorem then states that after correcting erasures, the probability of an uncorrected block at the final level is

$$q_m \leq m^4 n_1^2 \frac{1}{2} 6^{-3/2(3^{m-1}-1)} q_0^{3^m} e^{3 \sum_{k=1}^{m-1} 3^{m-k} \ln(k^2 n_1)}.$$

Upon estimating the sum in the exponent with Jensen's inequality,

$$2 \sum_{k=1}^{m-1} 3^{-k} \ln k \leq 2 \sum_{k=1}^{\infty} 3^{-k} \ln k \leq \ln \frac{3}{2},$$

it follows that

$$q_m \leq m^4 n_1^2 \frac{1}{2} 6^{-3/2(3^{m-1}-1)} q_0^{3^m} e^{3/2(3^m-1) \ln n_1} e^{3^{m+1} \ln \frac{3}{2}}.$$

To achieve exponential decay of q in 3^m requires

$$-\frac{3}{2} \ln 6 + 3 \ln q_0 + \frac{9}{2} \ln n_1 + 9 \ln \frac{3}{2} < 0,$$

which amounts to

$$\frac{27}{8\sqrt{6}} q_0 n_1^{3/2} < 1.$$

Since $n_1 = 3$ is the smallest dimension to start the iteration, feasible error correction needs $q_0 < 8\sqrt{2}/81 \approx 0.14$.

The number of transmitted frame coefficients is $(m!)^2 n_1^m$, so by Stirling's approximation $O(e^{(m+\frac{1}{2}) \ln m})$, whereas the decay of the mean-square error is of order $O(e^{-c3^m})$, for a suitable $c > 0$. This implies that q_m decays faster than any inverse power of the number of transmitted coefficients. Consequently, the same is true for the mean-square error.

5. CONCLUSIONS

In this paper we discussed the design of frames for transmitting vectors through a memoryless analog erasure channel, which transmits the frame coefficients perfectly or discards them, depending on the outcomes of Bernoulli trials with sufficiently small failure probability. We presented the construction of a nested sequence of frames which encode at a decreasing, asymptotically non-zero rate and allow the receiver to recover part of the erased coefficients. Finally, we showed that the mean-square reconstruction error remaining after corrections decays faster than any inverse power of the number of frame coefficients.

The interplay of coding theory and frame theory, i.e., the binary view-point and the analog view-point, respectively, is a general theme in our methodology and analysis. It allowed us to consider methods reminiscent of coding theory for analog data and to analyze the mean-square error resulting for the error correction algorithm and the class of fusion frames we considered.

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