FUSION FRAMES AND G-FRAMES IN BANACH SPACES

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Abstract. Fusion frames and g-frames in Hilbert spaces are generalizations of frames, and frames were extended to Banach spaces. In this article we introduce fusion frames, g-frames, Banach g-frames in Banach spaces and we show that they share many useful properties with their corresponding notions in Hilbert spaces. We also show that g-frames, fusion frames and Banach g-frames are stable under small perturbations and invertible operators.

1. Introduction and Preliminaries

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [11], later Daubechies, Grossmann and Meyer [10] found a fundamental new application. Nice properties of frames make them very useful in filter banks, sigma-delta quantization, signal and image processing. This notion was generalized to g-frames by Sun [16], see also [14, 15]. Fusion frames were introduced by Casazza et al in [6] and studied by many authors see [1, 4, 13]. Fusion frames have some applications in (wireless) sensor networks, see [5, 7]. Meanwhile frames were extended to Banach spaces by Grochenig [12] and studied in [2, 3].

In this article we introduce fusion frames, g-frames and Banach g-frames in Banach spaces. First we recall the following definitions from [12].

Definition 1.1. Let X_d be a Banach space of scalar-valued sequences. It is called a *BK-space* if the coordinate functionals are continuous on X_d and it is *solid* if whenever $\{b_i\}$ and $\{c_i\}$ are sequences with $\{c_i\} \in X_d$ and $|b_i| \leq |c_i|$, then it follows that $\{b_i\} \in X_d$ and $\|\{b_i\}\| \leq \|\{c_i\}\|$.

We note that $l^2(I)$ is a solid BK-space and if X_d is a solid BK-space such that for each $i \in I$, there exists some $x = \{x_j\}$ in X_d such that $x_i \neq 0$, then every $e_i = \{\delta_{ij}\}_{j\in I}$ is in X_d.

Definition 1.2. Let X be a Banach space with dual space X^* and X_d be a BK-space. A countable family $\{g_i\}$ in X^* is called an *X_d-frame* for X if there exist constants A,$B > 0$ such that for every $x \in X$, $\{g_i(x)\} \in X_d$ and $A\|x\| \leq \|\{g_i(x)\}\| \leq B\|x\|$.

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A and B are called X_d-frame *bounds*. If moreover there exists a bounded linear operator $S : X_d \to X$ such that for every $x \in X$, $S(\{g_i(x)\}_{i\in I}) = x$, then $\big(\{g_i\}_i, S\big)$ is a *Banach frame* for X with respect to X_d.

If there exists a sequence $\{x_i\}$ in X such that $x = \sum g_i(x)x_i$,for each $x \in X$, then $(\{g_i\}, \{x_i\})$ is an atomic decomposition of X with respect to X_d.

Definition 1.3. Let X be a Banach space with dual space X^*. A sequence $\{x_i\}$ in X is a *Schauder frame* for X if there exist a BK-space X_d and a sequence $\{g_i\}$ in X^* such that $(\{g_i\}, \{x_i\})$ is an atomic decomposition of X with respect to X_d. We also call the bounds of $\{g_i\}$, the bounds of $\{x_i\}$.

In Section 2 we study g-frames in Banach spaces and in Section 3 we study fusion frames in Banach spaces. Throughout this article X is a Banach space with dual space X^* and $\mathbb{N}$, $\mathbb{C}$ will denote the set of natural numbers and the field of complex numbers, respectively. I, J and I_i's will also denote a finite set or a subset of integers.

2. G-Frames

In this section we introduce g-frames in Banach spaces and we generalize some of their known results in Hilbert spaces to Banach spaces. Throughout this section X, Y are Banach spaces and $\{Y_i : i \in I\}$ is a sequence of closed subspaces of Y, as usual $B(X, Y_i)$ denotes the space of all bounded operators from X to Y_i and $B(X, X)$ is abbreviated by $B(X)$.

Definition 2.1. We call a sequence $\{\Lambda_i \in B(X, Y_i) : i \in I\}$ a g-frame for X with respect to $\{Y_i\}$ if there exists a solid BK-space X_d and positive constants $0 < A \le B < \infty$ such that

(i) for every $f \in X$, $\{\|\Lambda_i f\|\} \in X_d$,

(ii) for every $f \in X$, $A\|f\|_X \le \|\{\|\Lambda_i f\|\}\|_{X_d} \le B\|f\|_X$.

A and B are called *bounds of g-frame* and we say that $\{\Lambda_i\}$ is an (A, B)-g-frame.

We note that every X_d-frame $\{g_i\}$ is a g-frame for X, where $Y_i = \mathbb{C}$ for each $i \in I$. The following result is essential for our next investigations.

Theorem 2.2. *Let $\{\Lambda_i \in B(X, Y_i) : i \in I\}$ be a g-frame for X with respect to $\{Y_i : i \in I\}$. Then $W = (\oplus Y_i)_d = \{\{y_i\} : y_i \in Y_i, \{\|y_i\|\} \in X_d\}$,with $\|.\|_W$ defined by $\|\{y_i\}\|_W = \|\{\|y_i\|\}_{x_d}$ is a Banach space with coordinatewise operations and $\theta : X \to W$, the analysis operator, defined by $\theta(x) = \{\Lambda_i(x)\}$ is a bounded , one-to-one, linear operator with closed range and $\theta^{-1} : \theta(X) \to X$ is bounded.*

Proof. For every $w = \{w_i\}$, $z = \{z_i\}$ in W and every $i \in I$ we have $\|z_i + w_i\| \le \|z_i\| + \|w_i\|$. Since $\{\|w_i\|\}, \{\|z_i\|\} \in X_d$ and X_d is a solid space, then $\{\|z_i + w_i\|\} \in X_d$ and so $z + w \in W$, moreover

$$\|z + w\|_W = \|\{\|z_i + w_i\|\}\|_{X_d} \le \|\{\|z_i\|\}\|_{X_d} + \|\{\|w_i\|\}\|_{X_d} = \|z\|_W + \|w\|_W.$$

Plainly for each $\lambda \in \mathbb{C}$, $\|\lambda z\|_W = |\lambda| \cdot \|z\|_W$. So W is a normed space. For the completeness of W, we consider the map $\eta : W \to X_d$ defined by

$$\eta(\{y_i\}) = \{\|y_i\|\}, \qquad \text{for all } \{y_i\}_i \in W.$$

Firstly for each $w = \{w_i\}_i$ in W, we have $\|\eta(w)\|_{X_d} = \|w\|_W$. Secondly for each $i \in I$, $\|z_i\| - \|w_i\| \leq \|z_i - w_i\|$ and X_d is a solid space, then

$$\|\eta(z) - \eta(w)\|_{X_d} \leq \|z - w\|_W \quad \text{for all } z, w \in W. \tag{1}$$

Similarly by the continuity of coordinate functionals $\pi_i : X_d \to \mathbb{C}$, for each $i \in I$, the projection $P_i : W \to Y_i$ defined by

$$P_i(w) = w_i, \quad \text{for all } w = \{w_i\}_i \text{ in } W$$

is a continuous linear map, because

$$\|P_i(w)\| = \|w_i\| = \pi_i(\eta(w)) \leq \|\pi_i\| \cdot \|\eta(w)\| = \|\pi_i\| \cdot \|w\|. \tag{2}$$

Now let $\{w^{(n)}\}$ be a Cauchy sequence in W, where $w^{(n)} = \{w_i^{(n)}\}_{i\in I}$ for each $n \in \mathbb{N}$. By (2) for each $i \in I$, $\{w_i^{(n)}\}_n$ is a Cauchy sequence in Y_i and Y_i is complete. So there exists $w_i \in Y_i$ such that $w_i^{(n)} \to w_i$ as $n \to \infty$. Therefore for each $i \in I$, $\|w_i^{(n)}\| \to \|w_i\|$ as $n \to \infty$.

On the other hand by (1), $\{\eta(w^{(n)})\}_n$ is a Cauchy sequence in X_d and X_d is complete, so there exists some $\Lambda = \{\lambda_i\}_{i\in I}$ in X_d such that $\{\eta(w^{(n)})\}_n$ converges to Λ. Hence by the continuity of π_i's, for each $i \in I$, $\|w_i^{(n)}\| \to \lambda_i$ as $n \to \infty$. Therefore for each $i \in I$, $\lambda_i = \|w_i\|$ and $\Lambda = \{\|w_i\|\}_i$ is in X_d. So $w = \{w_i\}$ is in W.

Now for each $n \in \mathbb{N}$, $w^{(n)} - w \in W$ and $\{w^{(n)} - w\}_n$ is a Cauchy sequence in W and for each $i \in I$, $w_i^{(n)} - w_i \to 0$ as $n \to \infty$. Hence by the above proof $\{\eta(w^{(n)} - w)\}_n$ converges to 0 in X_d. Since $\|w^{(n)} - w\|_W = \|\eta(w^{(n)} - w)\|_{X_d}$, then $\{w^{(n)}\}$ converges to w in W. Hence W is complete.

For the second assertion for each $x \in X$,

$$A\|x\|_X \leq \|\theta(x)\|_W \leq B\|x\|_X,$$

then θ is bounded, one-to-one and θ^{-1} is bounded. Moreover since W is complete, θ has a closed range. □

Corollary 2.3. *With the hypothesis of Theorem 2.2 every Y_i is isomorphic to its copy E_i in W, where $E_i = \{\{y_i\delta_{ij}\}_{j\in I} : y_i \in Y_i\}$.*

Proof. First we show that E_i is a closed subspace of W. Since X_d is a solid space, then for each $y_i \in Y_i$, we have $\{\|y_i\|\delta_{ij}\}_{j\in I} \in X_d$ and therefore $\{y_i\delta_{ij}\}_{j\in I} \in W$, which implies that $E_i \subseteq W$. Plainly E_i is a subspace of W. If $\{u^{(n)}\}_{n\in\mathbb{N}}$ is a sequence in E_i converges to $z = \{z_j\}_{j\in I}$ in W, where $u^{(n)} = \{y_i^{(n)}\delta_{ij}\}_{j\in I}$, then for each $j \in I$,

$$\|y_i^{(n)}\delta_{ij} - z_j\| = \pi_j(\eta(u^{(n)} - z)) \leq \|\pi_j\| \cdot \|\eta(u^{(n)} - z)\|_{X_d} \leq \|u^{(n)} - z\|_W.$$

Therefore $y_i^{(n)}\delta_{ij} \to z_j$. Specially for $j = i$, we get $y_i^{(n)} \to z_i$ and since Y_i is closed, $z_i \in Y_i$. If $j \neq i$, then $y_i^{(n)}\delta_{ij} = 0$ and consequently $z_j = 0$. Hence $z = \{z_i\delta_{ij}\}$ is in E_i which shows that E_i is closed. Now the coordinate projection $P_i : E_i \to Y_i$ is a continuous bijection, since $\|P_i(\{y_i\delta_{ij}\}_j)\| = \|\pi_i\eta(y)\| \leq \|\pi_i\| \cdot \|y\|$. Therefore by the open mapping theorem it is an isomorphism. □

For our next investigation we need the following result, and for convenience we abbreviate $\|.\|_{Z_{i,d}}$ by $\|.\|_i$.

Corollary 2.4. *Let for each $i \in I$, $(Z_{i,d}, \|.\|_i)$ be a BK-space and X_d be a solid BK-space. Then*

$$S_d = (\oplus_{i\in I} Z_{i,d}) = \{\{\Lambda_i\}_{i\in I} : \ \Lambda_i = \{\lambda_{i,j} : \ j \in I_i\} \in Z_{i,d}, \ \{\|\Lambda_i\|_i\}_{i\in I} \in X_d\}$$

with $\|.\|_{S_d}$ defined by $\|\{\Lambda_i\}_i\|_{S_d} = \|\{\|\Lambda_i\|_i\}_{i\in I}\|_{X_d}$ is a BK-space and if moreover each $Z_{i,d}$ is solid, then S_d is solid.

Proof. By the above theorem S_d is a Banach space and for each $i \in I, j \in I_i$, the coordinate functional $\{\lambda_{ij} : \ j \in I_i\}_{i\in I} \to \lambda_{ij}$ is $\pi_{ij} \circ P_i$ which is continuous, because X_d and each $Z_{i,d}$ is a BK-space. Moreover suppose that each $Z_{i,d}$ is a solid space and let for each $i \in I$, $j \in I_i$, $|b_{ij}| \leq |\lambda_{ij}|$ and $\{\lambda_{ij}\} \in S_d$. Let $i \in I$. Since $\Lambda_i = \{\lambda_{ij} : \ j \in I_i\}$ is in $Z_{i,d}$ and $Z_{i,d}$ is solid, then $\{b_{ij} : \ j \in I_i\}$ is in $Z_{i,d}$ and

$$\|\{b_{ij} : \ j \in I_i\}\|_i \leq \|\{\lambda_{ij} : \ j \in I_i\}\|_i.$$

Now $\{\|\{\lambda_{ij} : \ j \in I_i\}\|_i\}_i \in X_d$ and X_d is solid, so by the above relation $\{\|\{b_{ij} : \ j \in I_i\}\|_i\}_i \in X_d$. Therefore $\{b_{ij} : \ j \in I_i\}_i \in S_d$ and moreover

$$\begin{aligned} \|\{\{b_{ij} : \ j \in I_i\}\}_i\| &= \|\{\|\{b_{ij} : \ j \in I_i\}\|_i\}_i\|_{X_d} \\ &\leq \|\{\|\{\lambda_{ij} : \ j \in I_i\}\|_i\}_i\|_{X_d} = \|\{\{\lambda_{ij} : \ j \in I_i\}_i\|. \end{aligned}$$

□

Our next result is a generalization of Theorem 3.4 of [7] and Theorem 2.2 of [15].

Theorem 2.5. *Let $\{\Lambda_i \in B(X, Y) : i \in I\}$ and let for each $i \in I$, $\{\Gamma_{ij} \in B(Y_i, W_{ij}) : j \in I_i\}$ be an (A_i, B_i)-g-frame for Y_i with respect to $\{W_{ij} : j \in I_i\}$ and associated BK-space $Z_{i,d}$ and suppose that* $0 < A = \inf_i A_i \leq B = \sup_i B_i < \infty$. *Then the following conditions are equivalent:*

(i) $\{\Lambda_i \in B(X, Y) : i \in I\}$ is a g-frame for X with associated BK-space X_d.

(ii) $\{\Gamma_{ij}\circ\Lambda_i \in B(X, W_{ij}) : i \in I, j \in I_i\}$ is a g-frame for X with associated BK-space S_d.

Proof. First note that for every $x \in X$ and every $i \in I$, we have $\Lambda_i x \in Y_i$, so $\{\|\Gamma_{ij} \circ \Lambda_i(x)\| : \ j \in I_i\} \in Z_{i,d}$ and

(3)
$$A\|\Lambda_i x\| \leq A_i\|\Lambda_i x\| \leq \|\{\|\Gamma_{ij} \circ \Lambda_i(x)\| : j \in I_i\}\|_i \leq B_i\|\Lambda_i(x)\| \leq B\|\Lambda_i x\|.$$

To prove $(i) \Rightarrow (ii)$ let $\{\Lambda_i \in B(X, Y_i); i \in I\}$ be a g-frame with bounds C and D. Then for every $x \in X$, $\{\|\Lambda_i x\|\} \in X_d$ and

$$C\|x\|_X \leq \|\{\|\Lambda_i x\|\}_i\|_{X_d} \leq D\|x\|. \tag{4}$$

Since $\{\|\Lambda_i x\|\} \in X_d$ and X_d is solid, then by (3) we have

$$\{\|\{\|\Gamma_{ij} \circ \Lambda_i(x)\| : \ j \in I_i\}\|_i\} \in X_d$$

and $\|\{\|\{\|\Gamma_{ij} \circ \Lambda_i(x)\| : \ j \in I_i\}\|_i\}\|_{X_d} \leq B\|\{\|\Lambda_i(x)\|\}_i\|_{X_d}$. Hence by (4) we conclude that $\|\{\|\{\|\Gamma_{ij} \circ \Lambda_i(x)\| : \ j \in I_i\}\|_i\}\|_{X_d} \leq BD\|x\|_X$. Similarly we get $AC\|x\|_X \leq \|\{\|\{\|\Gamma_{ij} \circ \Lambda_i(x)\| : \ j \in I_i\}\|_i\}\|_{X_d}$.

To prove $(ii) \Rightarrow (i)$ let $\{\Gamma_{ij} \circ \Lambda_i \in B(X, W_{ij}) : i \in I, j \in I_i\}$ be a (C', D')-g-frame with associated BK-space S_d. Then for each $x \in X$, $\{\{\Gamma_{ij} \circ \Lambda_i(x) : j \in I_i\}\}_i \in S_d$,

$$\|\{\{\Gamma_{ij} \circ \Lambda_i(x) : \ j \in I_i\}\}_i\|_{S_d} = \|\{\|\{\|\Gamma_{ij} \circ \Lambda_i(x)\| : \ j \in I_i\}\|_i\}_i\|_{X_d}$$

and $C'\|x\|_X \leq \|\{\{\Gamma_{ij} \circ \Lambda_i(x) : \ j \in I_i\}\}_i\|_{S_d} \leq D'\|x\|_X$. Again by (3) and the solidity of X_d we conclude that $\{\|\Lambda_i x\|\} \in X_d$ and $A\|\{\|\Lambda_i x\|\}_i\|_{X_d} \leq D'\|x\|_X$. Similarly we get $C'\|x\|_X \leq B\|\{\|\Lambda_i x\|\}_i\|_{X_d}$, which completes the proof. □

If $\{\Lambda_i \in B(X, Y_i) : \ i \in I\}$ is a g-frame for X with respect to $\{Y_i : \ i \in I\}$ and there exists a bounded linear map $\eta : W \to X$ such that for each $x \in X$, $\eta(\{\Lambda_i(x)\}_i) = x$, we call $(\{\Lambda_i\}, \eta)$ a *Banach g-frame.* If $\{\Lambda_i \in B(X, Y_i) : i \in I\}$ is a g-frame and there exists a sequence $\{q_i \in B(Y_i, X) : \ i \in I\}$ such that for each $x \in X$, $x = \sum_i q_i \circ \Lambda_i(x)$, then $(\{\Lambda_i\}, \{q_i\})$ is called a (linear) *decomposition* of X with respect to $\{Y_i : \ i \in I\}$.

We note that if $(\{\Lambda_i\}, \eta)$ is a Banach g-frame, then $\eta \circ \theta = \mathrm{id}_X$ and $p = \theta \circ \eta$ is a projection which implies that $\theta(X) = p(W)$ is a complemented subspace of W.

From the above theorem we have the following consequence.

Corollary 2.6. *Let X be a Banach space and $\{\Lambda_i \in B(X, Y_i) : \ i \in I\}$ be a g-frame. Let $\theta : X \to W$ be the analysis operator of $\{\Lambda_i \in B(X, Y_i) : i \in I\}$. Then $(\{\Lambda_i\}, \theta^{-1})$ is a Banach g-frame.*

Our next result is a generalization of Lemma 2.13 in [13] to Banach spaces.

Lemma 2.7. *Let $\{\Lambda_i \in B(X, Y_i)\}_i$ be a g-frame with bounds A, B and let $T_i \in B(Y_i, Z_i)$ be a bounded invertible operator. Suppose that*

$$0 < m = \inf_i \frac{1}{\|T_i^{-1}\|} \leq \sup_i \|T_i\| = M < \infty.$$

If $T \in B(X)$ is invertible and $\Gamma_i = T_i \Lambda_i T$, then $\{\Gamma_i \in B(X, Z_i)\}$ is a g-frame. If moreover $(\{\Lambda_i\}, \{q_i\})$ is a linear decomposition of X with respect to $\{Y_i : \ i \in I\}$, then there exists a sequence $\{P_i \in B(Z_i, X) : \ i \in I\}$ such that $(\{\Gamma_i\}, \{P_i\})$ is a linear decomposition of X with respect to $\{Z_i : \ i \in I\}$.

Proof. Let $f \in X$. For each $j \in I$ we have

$$m\|\Lambda_j(Tf)\| \leq \|\Gamma_j f\| \leq M\|\Lambda_i(Tf)\|.$$

Since X_d is a solid space and $\{\|\Lambda_j T_f\|\} \in X_d$, then $\{\|\Gamma_j f\|\} \in X_d$ and $\|\{\|\Gamma_j f\|\}\|_{X_d} \leq M\|\{\|\Lambda_j Tf\|\}\|_{X_d}$, similarly $m\|\{\|\Lambda_j Tf\|\}\|_{X_d} \leq \|\{\|\Gamma_j f\|\}\|_{X_d}$. Hence for every $f \in X$,

$$\begin{aligned} mA\frac{1}{\|T^{-1}\|} \cdot \|f\|_X &\leq mA\|Tf\|_X \leq m\|\{\|\Lambda_j Tf\|\}_j\|_{X_d} \\ &\leq \|\{\|\Gamma_j f\|\}_j\|_{X_d} \leq MB\|Tf\| \leq MB\|T\| \cdot \|f\|. \end{aligned}$$

If $(\{\Lambda_i\}, \{q_i\})$ is a linear decomposition of X, then by taking $p_i = T^{-1}q_i T_i^{-1}$ we see that for every $f \in X$, $\sum T^{-1}q_i T_i^{-1}(T - i\Lambda_i T)(f) = T^{-1}\sum q_i \Lambda_i Tf = f$ and we have the result. □

Christensen and Heil in [9] showed that Banach frames are stable under small perturbations. We show that g-frames and Banach g-frames are also stable under small perturbations.

Definition 2.8. Let $\{\Lambda_i \in B(X, Y_i) : \ i \in I\}$ be a g-frame and let $0 \leq \lambda, \mu < 1$. Let $\{c_i\} \in X_d$. We say that $\{\Gamma_i \in B(X, Y_i) : \ i \in I\}$ is a $(\lambda, \mu, \{c_i\})$-*perturbation* of $\{\Lambda_i\}$ if

$$\|\Lambda_i f - \Gamma_i f\| \leq \lambda\|\Lambda_i f\| + \mu\|\Gamma_i f\| + c_i\|f\|, \quad \text{for each } f \in X.$$

Now we have the following result, see [15, Proposition 4.3].

Theorem 2.9. *Let $\{\Lambda_i \in B(X, Y_i) : \ i \in I\}$ be a g-frame for X with respect to $\{Y_i : \ i \in I\}$ with bounds A, B and let $\{\Gamma_i \in B(X, Y_i) : \ i \in I\}$ be a $(\lambda, \mu, \{c_i\})$-perturbation of it with $K = \|\{c_i\}\|_{X_d}$.*

(a) If $A' = (1-\lambda)A(1+\mu)^{-1} - K > 0$, then $\{\Gamma_i\}$ is a g-frame with bounds A' and $B' = (1+\mu)(1+\lambda)B(1-\mu)^{-1} + K(1+\mu)(1-\mu)^{-1}$.

(b) If $(\{\Lambda_i\}, \eta)$ is a Banach g-frame, $A' > 0$, and $\|\eta\|(B\lambda + \mu B' + K) < 1$, then there exists $T \in B(X)$ such that $(\{\Gamma_i\}, T)$ is a Banach g-frame.

Proof. (a) By assumption for each $f \in X$ we have $\|\Lambda_i f - \Gamma_i f\| \leq \lambda\|\Lambda_i f\| + \mu\|\Gamma_i f\| + c_i\|f\|$. Hence

$$(1-\mu)\|\Gamma_i f\| \leq (1+\lambda)\|\Lambda_i f\| + c_i\|f\|.$$

Now $\{\|\Lambda_i f\|\}_i \in X_d$, $\{c_i\} \in X_d$ and X_d is a solid space, therefore $\{\|\Gamma_i f\|\} \in X_d$ and moreover

$$(1-\mu)\|\{\|\Gamma_i f\|\}\|_{X_d} \leq (1+\lambda)\|\{\|\Lambda_i f\|\}\|_{X_d} + \|f\|K.$$

Similarly we have

$$(1-\lambda)\|\{\|\Lambda_i f\|\}\|_{X_d} \leq (1+\mu)\|\{\|\Gamma_i f\|\}\|_{X_d} + \|f\|K.$$

Now a simple calculation shows that

$$A'\|f\|_X \leq \|\{\|\Gamma_i f\|\}\|_{X_d} \leq B'\|f\|_X, \quad \text{for each } f \in X$$

which completes the proof of (a).

(b) In (a) we proved that for each $f \in X$, $\{\|\Gamma_i f\|\} \in X_d$ and therefore $\{\Gamma_i f\} \in W$. So we can define $S : X \to W$ by $S(f) = \{\Gamma_i f\}$. Plainly S is a linear operator and by the above relation $\|S\| \leq B'$. Moreover for every $f \in X$,

$$\begin{aligned} \|\eta \circ S(f) - f\| = \|\eta(S - \theta)(f)\| &\leq \|\eta\| \cdot \|S(f) - \theta(f)\|_{X_d} \\ &\leq \|\eta\|(\lambda B + \mu B' + K)\|f\|. \end{aligned}$$

Hence $\|\eta \circ S - I\| \leq \|\eta\|(\lambda B + \mu B' + K) < 1$, therefore $\eta \circ S$ is invertible in $B(X)$ and for $T = (\eta \circ S)^{-1} \circ \eta$ we have $TS = I$, which completes the proof of (b). □

3. Fusion Frames

Fusion frames in Hilbert spaces were introduced by Casazza et al. in [2], [3], it has been intensely studied and some of its applications are discovered [3] and [6]. In this section we introduce fusion frames in Banach spaces and we discuss some of their properties.

Definition 3.1. Let X be a Banach space, $\{p_i\}_{i\in I}$ be a sequence of continuous linear projections on X, $W_i = p_i(X)$ for each $i \in I$ and let $\{v_i : i \in I\}$ be a sequence of weights, i.e., $v_i > 0$. We say that $\{(v_i, W_i) : i \in I\}$ is a *fusion frame* for X if there exists a sequence $\{q_i \in B(W_i, X) : i \in I\}$ and an invertible operator $S \in B(X)$ such that $(\{v_i S^{-1} \circ q_i\}, \{v_i p_i\})$ is an atomic decomposition of X with respect to $\{W_i : i \in I\}$.

We note that if $\{(v_i, W_i) : i \in I\}$ is a fusion frame for X with associated sequence $\{q_i \in B(W_i, X) : i \in I\}$, then for each $x \in X$, the series $\sum_{i\in I} v_i^2 q_i \circ p_i(x)$ converges to $S(x)$ and there exist a solid BK-space X_d and constants $0 < A \leq B$ such that for every $x \in X$,

$$A\|x\|_X \leq \|\{\|p_i(x)\|\}\|_{X_d} \leq B\|x\|.$$

The constants A and B are called the fusion frame bounds of $\{(v_i, W_i) : i \in I\}$, it is called tight if $A = B$ and Parseval if $A = B = 1$. Also note that if $\{(v_i, W_i) : i \in I\}$ is a fusion frame, where $W_i = p_i(X)$, then $\{v_i p_i \in B(X, W_i) : i \in I\}$ is a g-frame.

Lemma 3.2. *Let X be a Banach space and $\{f_i : i \in I\}$ be a Schauder frame for X with bounds A and B. Then $\{\|f_i\|^{-1}, Span\{f_i\})_{i\in I}$ is a fusion frame with bounds A and B.*

Proof. Since $\{f_i : i \in I\}$ is a Schauder frame with bounds A and B, there exists a sequence $\{g_i : i \in I\}$ in X^* and a solid BK-space X_d such that for each $x \in X$, $x = \sum g_i(x) f_i$ and $A\|x\|_X \leq \|\{g_i(x)\}\|_{X_d} \leq B\|x\|_X$.

In a Banach space every finite dimensional subspace is closed and complemented, so $W_i = span\{f_i\}$ is closed, complemented and its associated continuous projection $p_i : X \to W_i$ is defined by

$$p_i(f) = g_i(f) f_i \qquad \text{for all } f \in X,$$

and

$$A\|f\|_X \leq \|\{v_i p_i(f)\}\|_{X_d} \leq B\|f\|_X,$$

where $v_i = 1/\|f_i\|$. We can take q_i the identity map on X for each $i \in I$. $\square$

The following result shows that our definition consistent with the definition in [7].

Example 3.3. Every fusion frame $\{(v_i, W_i) : i \in I\}$ in a Hilbert space H is a fusion frame in H as a Banach space. To see this, we know that there exist constants A and B such that for each $x \in H$,

$$A\|x\|_H^2 \leq \sum_i v_i^2 \|\pi_{W_i}(x)\|^2 \leq B\|x\|^2$$

and S_W, the fusion frame operator of $\{(v_i, W_i) : i \in I\}$ is defined by

$$S_W(f) = \sum_i v_i^2 \pi_{W_i}(f) \quad \text{for all } f \in H,$$

is invertible. So $\{(v_i, W_i) : i \in I\}$ is a fusion frame for the Banach space $X = H$, where $p_i = \pi_{W_i} = q_i$ and $X_d = \ell^2(I)$.

For constructing g-frames, from Theorem 2.5 we have the following results for fusion frames and g-frames.

Lemma 3.4. *Let $\{\Lambda_i \in B(X, Y_i) : i \in I\}$ be a sequence of bounded operators and let for each $i \in I$, $\{(\alpha_{ij}, W_{ij}) : j \in I_i\}$ be a fusion frame for Y_i with bounds A_i, B_i and suppose that $0 < A = \inf_i A_i \leq B = \sup_i B_i < \infty$. Then the following conditions are equivalent:*

(i) $\{\Lambda_i\}_{i \in I}$ is a g-frame for X with respect to $\{Y_i : i \in I\}$,

(ii) $\{\alpha_{ij}\pi_{ij} \circ \Lambda_i : i \in I, j \in I_i\}$ is a g-frame for X with respect to $\{W_{ij} : i \in I, j \in I_i\}$, where $\pi_{i_j} : Y_i \to W_{ij}$ is the projection of Y_i onto W_{ij}.

Proof. It is enough to note that $\{\alpha_{ij}\pi_{ij} : j \in I_I\}$ is a g-frame for W_{ij} with bounds A_i, B_i and use Theorem 2.5. $\square$

Another version of these combinations is as follows:

Corollary 3.5. *Let $\{(\alpha_i, Y_i) : i \in I\}$ be a fusion frame with bounds C and D. Let for each $i \in I$, $\{\Lambda_{ij} \in B(Y_i, W_{ij}) : j \in I_i\}$ be an (A_i, B_i)-g-frame for Y_i, such that $0 < A = \inf_i A_i \leq B = \sup_i B_i < \infty$. Then $\{\alpha_i \Lambda_{ij} \circ \pi_{Y_i} : i \in I, j \in I_i\}$ is an (AC, BD)-g-frame.*

Proof. It is enough to note that $\{\alpha_i \pi_{Y_i} : i \in I\}$ is a (C, D)-g-frame for X and use Theorem 2.5. $\square$

Theorem 3.6. *Let $\{(v_i, W_i) : i \in I\}$ be a fusion frame for X with bounds C, D; associated sequence $\{q_i \in B(W_i, X) : i \in I\}$ and let $\{f_{ij}\}_{j \in I_i}$ be a Schauder frame for W_i with bounds A_i, B_i for each $i \in I$. If $0 < A = \inf_i A_i \leq B = \sup_i B_i < \infty$, then $\{\{S^{-1}q_i(f_{ij})\}_{j \in I_i}\}_{i \in I}$ is a Schauder frame for X with bounds AC and BD. In particular if $\{(v_i, W_i) : i \in I\}$ and $\{f_{ij} : j \in I_i\}$ are Parseval, then $\{\{S^{-1}q_i(f_{ij}\}_j\}_i$ is Parseval.*

Proof. Since $\{(v_i, W_i) : i \in I\}$ is a fusion frame with associated sequence $\{q_i \in B(W_i, X) : i \in I\}$, then there exists an invertible linear operator S in $B(X)$ such that for each $x \in X$, $x = \sum_i v_i^2 S^{-1} \circ q_i \circ p_i(x)$, where p_i is the projection of X onto W_i. We also know that for each $i \in I$, $\{f_{ij}\}_{j \in I_i}$ is a Schauder frame for W_i, so there exists a sequence $\{g_{ij}\}_{j\in I_i} \subseteq W_i^*$ and a solid BK-space $Z_{i,d}$ such that for each $x \in X$, $\{g_{ij} \circ p_i(x)\}_{j \in I_i} \in Z_{i,d}$ and $p_i(x) = \sum_{j \in I_i} g_{ij}(p_i(x)) f_{ij}$. Therefore for each $x \in X$ we have

$$x = \sum_{i\in I} \sum_{j \in I_i} g_{ij}(p_i(x)) S^{-1} q_i(f_{ij}). \tag{5}$$

Now a small modification in the proof of Theorem 2.5 shows that for each $x \in X$, $\{\{v_i g_{ij} \circ p_i(x)\}_{j\in I_i}\}_{i \in I} \in S_d$ and

$$AC\|x\|_X \le \|\{\|\{v_i g_{ij} \circ p_i(x)\}_j\|_i\}_i\|_{X_d} \le BD\|x\|_X.$$

Since $\{\{v_i g_{ij} \circ p_i\}_{j\in I_i}\}_{i\in I} \subseteq X^*$, S_d is a solid BK-space and (5) holds for each $x \in X$, then $\{\{S^{-1} q_i(f_{ij}\}_{j\in I_i}\}_{i \in I}$ is a Schauder frame for X. □

Proposition 3.7. *Let $\{(v_i, W_i) : i \in I\}$ be a fusion frame for X and $T \in B(X)$ be invertible. Then $\{(v_i, TW_i) : i \in I\}$ is a fusion frame for X.*

Proof. Since $\{(v_i, W_i) : i \in I\}$ is a fusion frame for X, then each W_i is complemented in X, there exist a solid BK-space X_d, constants $0 < A \le B < \infty$, a sequence $\{q_i \in B(W_i, X) : i \in I\}$ and an invertible operator $S \in B(X)$ such that for each $x \in X$, $\{v_i\|p_i(x)\| : i \in I\} \in X_d$, $x = \sum_i v_i^2 S^{-1} \circ q_i \circ p_i(x)$ and $A\|x\|_X \le \|\{v_i\|p_i(x)\|\}_i\|_{X_d} \le B\|x\|_X$, where $p_i = \pi_{W_i}$ is the projection of X onto W_i.

Now since $T \in B(X)$ is invertible and W_i is complemented, then TW_i is complemented in X and $\pi_{TW_i} = T p_i T^{-1}$, for each $i \in I$. Moreover for each $x \in X$ and $i \in I$, $\|T p_i T^{-1}(x)\| \le \|T\| \cdot \|p_i T^{-1}(x)\|$. Since $\{v_i\|p_i T^{-1}(x)\|\} \in X_d$ and X_d is a solid space, then $\{v_i\|T p_i T^{-1}(x)\|\}_i \in X_d$ and $\|\{v_i\|T p_i T^{-1}(x)\|\}_i\|_{X_d} \le \|T\| \cdot \|\{v_i\|p_i T^{-1}(x)\|\}_i\|_{X_d}$. Hence for each $x \in X$,

$$\frac{1}{\|T\|} \cdot \frac{1}{\|T^{-1}\|} A\|x\|_X \le \|\{v_i\|T p_i T^{-1}(x)\|\}_i\|_{X_d} \le \|T\| \cdot \|B\| \cdot \|T^{-1}\| \cdot \|x\|_X.$$

Finally $\{q_i T^{-1} \in B(TW_i, X)\}$, $ST^{-1} \in B(X)$ is invertible and for each $x \in X$, we have $x = \sum_i v_i^2 T(S^{-1} \circ q_i \circ T^{-1})(T \circ p_i \circ T^{-1})(x)$, which completes the proof. □

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