

# ADAPTIVE DIRECTIONAL SUBDIVISION SCHEMES AND SHEARLET MULTIREOLUTION ANALYSIS

GITTA KUTYNIOK AND TOMAS SAUER

ABSTRACT. In this paper, we propose a solution for a fundamental problem in computational harmonic analysis, namely, the construction of a multiresolution analysis with directional components. We will do so by constructing subdivision schemes which provide a means to incorporate directionality into the data and thus the limit function. We develop a new type of non-stationary bivariate subdivision schemes, which allow to adapt the subdivision process depending on directionality constraints during its performance, and we derive a complete characterization of those masks for which these adaptive directional subdivision schemes converge. In addition, we present several numerical examples to illustrate how this scheme works. Secondly, we describe a fast decomposition associated with a sparse directional representation system for two dimensional data, where we focus on the recently introduced sparse directional representation system of shearlets. In fact, we show that the introduced adaptive directional subdivision schemes can be used as a framework for deriving a shearlet multiresolution analysis with finitely supported filters, thereby leading to a fast shearlet decomposition.

## 1. INTRODUCTION

Efficient and economical representations of anisotropic structures are essential in various areas in applied mathematics. The nature of the problems we face can be divided into two types, namely when the anisotropic structure is given explicitly and when it is given implicitly. The analysis of images and higher dimensional data with respect to directional features shall serve as an example of an explicitly given anisotropic structure, whereas the solution of hyperbolic partial differential equations often exhibits the phenomenon of shocks which can be interpreted as an implicit anisotropic structure.

It is well known that wavelets are perfectly suited for providing efficient representations in the sense of sparsity for problems with a dominant isotropic regularity, at the same time being associated with a multiresolution analysis which is the key ingredient for a fast decomposition algorithm. However, when dealing with anisotropic phenomena wavelets do not perform equally well. In fact, it can be proven that wavelets do not provide optimally sparse representations.

In contrast to earlier approaches such as directional wavelets [1], complex wavelets [21], ridgelets [3], and contourlets [16], the curvelets introduced by Candès and

---

*Date:* October 13, 2007.

*2000 Mathematics Subject Classification.* Primary 42C40; Secondary 41A05, 42C15, 47B99, 65D10, 94A08.

*Key words and phrases.* Directional transforms, joint spectral radius, multiresolution analysis, refinement equation, shearlets, subdivision schemes.

The first author was supported by Deutsche Forschungsgemeinschaft (DFG) Heisenberg-Fellowship KU 1446/8-1.

Donoho precisely satisfy this need, in the sense of resolving the wavefront set [5] and the curvelet representation being optimally sparse for objects with  $C^2$ -singularities [4]. Also there already exist some first results on applying curvelets to hyperbolic partial differential equations by Candès and Demanet [2]. However, one drawback is the lack of a multiresolution analysis associated with curvelets, and, in particular, a fast decomposition algorithm in the time domain. This raises the question about the existence of a representation system with analyzing properties as good as curvelets, but being equipped with a more “wavelet-like” structure in the sense of being associated with a multiresolution analysis. In fact, the discrete counterpart would then lead to finitely supported filters that allow for a mathematically justified discrete fast decomposition of discrete data. We anticipate such a representation to combine the favorable computational properties of wavelets with the main additional property to provide a means to resolve anisotropic structures efficiently.

In this paper we give a complete, positive answer to the question of the existence of such a system by introducing subdivision schemes for the recently introduced concept of shearlets, thus constructing an associated multiresolution analysis which indeed leads to a fast discrete decomposition algorithm. The directional representation system of *shearlets* [19] stands out for the following reason. They do not only precisely resolve the wavefront set [22] and provide optimally sparse representations [20], but shearlet systems are generated by one single function which is dilated by a parabolic scaling and a shear matrix and translated in the time domain, hence form an affine system. We might even interpret the system of shearlets as being generated by a strongly continuous, irreducible, square-integrable representation of a certain group, the shearlet group [11]. This rich mathematical structure enables, for instance, the application of coorbit theory to study smoothness spaces – so-called shearlet coorbit spaces – associated with the decay of the shearlet coefficients [12]. We would further like to mention that one attempt to associate shearlets with a so-called generalized multiresolution analysis can be found in [24]. However, this structure did not yield a fast decomposition due to the fact that the filters are not compactly supported and even infinitely many filters have to be employed.

Our approach to derive a multiresolution analysis associated with shearlets and to provide a feasible fast shearlet decomposition comprises the introduction of a new class of non-stationary bivariate subdivision schemes which incorporate directionality in a particular way. Subdivision schemes provide a mathematical method to refine given coarse data while providing characterization results to ensure convergence to a continuous function, say. Moreover, such schemes automatically provide *refinable functions* which are the basis for any multiresolution analysis as nestedness of the different levels of resolution is equivalent to the refinability of the underlying “basis” function. Homogeneous stationary subdivision schemes have been studied extensively over the last 20 years; for an elaborate survey we refer the reader to [6]. Recently, algebraic methods have been introduced as a means to derive characterizations of convergence and approximation order in a very natural way for multivariate subdivision (cf. [29]). On the other hand, also the conditions of homogeneity and stationarity have been relaxed by various authors, leading to subdivision schemes where the refinement rule varies with the level of iteration or the location of refinement. However, the gain in generality always comes with the prize of a loss of structure so that there is comparatively little known about these generalizations (see, e.g., [9, 7]). In particular, no subdivision schemes were known

so far which provide a means to adapt the subdivision process depending on directionality constraints during its performance while still ensuring convergence. The development of such subdivision schemes will be important both for construction of a shearlet multiresolution analysis as well as for opening the research area of methods for data refinement to incorporate anisotropic structures.

We will show in this paper that such an adaptive directional subdivision scheme can be constructed and it will indeed lead to a shearlet multiresolution analysis and a fast shearlet decomposition. Our approach to derive a non-stationary bivariate adaptive directional subdivision scheme is based on the idea to iteratively apply two subdivision schemes each of which is associated with a different direction. The two individual subdivision schemes can employ two different finitely supported filters while their respective dilation matrices are taken from the theory of shearlet systems. We would also like to mention at this point that the most natural “directional” operation, the rotation, can not be employed, since its action does not provide a refinement of a lattice. In contrast to this observation, products of parabolic scaling and shear matrices do indeed satisfy this desirable property. The constructed subdivision scheme provides the opportunity to adaptively change the orientation of the data during the subdivision process, since in each iteration one of both single subdivision schemes can be applied. In this sense, we can visualize the subdivision process as a binary tree, in which the direction of the finer data is dependent on the branch we choose. However, for convergence we certainly need to study each branch of the tree, which requires an appropriate definition of convergence. Our first key result shows that, provided the adaptive directional subdivision scheme converges, we obtain associated generalized refinement equations (Theorem 4.6). These will become essential for deriving a shearlet multiresolution analysis. As a main result we then provide a complete characterization of those masks which lead to convergent adaptive directional subdivision schemes (Theorem 4.14) in terms of algebraic and spectral properties of the associated filters. In the proof we will make use of ideal theoretic methods which come in handy to extract “the zero at  $-1$ ” of the two masks.

For the construction of a shearlet multiresolution analysis we employ the fact that each wavelet multiresolution analysis is associated with a convergent subdivision scheme [14]. We introduce scaling spaces based on the previously constructed directional subdivision schemes, and then prove that these indeed provide a multiresolution analysis structure (Theorem 6.3) due to the refinement equations mentioned above. This multiresolution analysis will then provide us in a very natural way with a mathematically justified discrete fast shearlet decomposition of discrete data which is stated as Algorithm 7.6. Also here we encounter a binary tree structure, since the decomposition will be dependent on the different directions which were encoded in a binary tree structure of the subdivision process. For the construction of a shearlet multiresolution analysis and a fast shearlet decomposition, we focus on the situation of interpolatory masks. The non-interpolatory case is beyond the scope of this paper and will be studied in a forthcoming paper.

The outline of the paper is the following. In Section 2 we briefly introduce discrete shearlet systems. We further study which directions can be attained by the action of the associated dilation matrices on  $\mathbb{Z}^2$ . The new type of subdivision schemes, which we baptize *adaptive directional subdivision schemes*, are introduced in Section 3. In Section 4 we provide a complete characterization of convergence for

those schemes along with the necessary ideal theoretic background. Some numerical experiments on the refinement of data employing this new type of subdivision schemes are provided in Section 5. We then show how the previously derived adaptive directional subdivision schemes can be used as a framework for deriving a *shearlet multiresolution analysis* with finitely supported filters (Section 6). In Section 7 we employ these results to provide a *fast shearlet decomposition*.

## 2. REFINEMENT OF $\mathbb{Z}^2$ BY ANISOTROPIC SCALING AND SHEARING

**2.1. Shearlet Dilation Matrices.** Our approach towards directional refinement of the lattice  $\mathbb{Z}^2$  and, later on, adaptive directional subdivision schemes is inspired by the recently introduced discrete shearlet transform [19], since this transform is able to precisely detect directions of singularities (cf. [22]) which we will take advantage of. In order to provide a thorough motivation for our construction, allow us to first briefly review the idea of shearlets.

Each shearlet system forms an affine system, i.e., consists of dilations and translations of one single generating function  $\psi \in L^2(\mathbb{R}^2)$ , a so-called *shearlet*. As dilation matrices, products of anisotropic parabolic scaling matrices and shear matrices – which coined the name “shearlets” – are employed. In order to define a shearlet system, let  $A_a$ ,  $a > 0$ , and  $S_s$ ,  $s \in \mathbb{R}$ , which are defined by

$$A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix} \quad \text{and} \quad S_s = \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix},$$

denote a *parabolic scaling matrix* and a *shear matrix*, respectively. Then the *shearlet system* associated with a shearlet  $\psi \in L^2(\mathbb{R}^2)$  is given by

$$(2.1) \quad \{\psi_{jkm}(x) := 2^{-\frac{3}{2}j}\psi(S_{-k}A_{4^{-j}}x - m) : j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}.$$

The three parameters  $j, k, m$  are interpreted in the following way:  $j$  provides the scale, and  $k$  and  $m$  detect the direction and position of singularities, respectively. It is easy to construct shearlets such that (2.1) forms a Parseval frame for  $L^2(\mathbb{R}^2)$ , for instance, by choosing  $\hat{\psi}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_1)\hat{\psi}_2(\xi_2/\xi_1)$ , where  $\psi_1 \in L^2(\mathbb{R})$  is a discrete wavelet, i.e.,  $\sum_{j \in \mathbb{Z}} |\hat{\psi}_1(4^j \omega)|^2 = 1$  for  $\omega \in \mathbb{R}$ , satisfying  $\hat{\psi}_1 \in C^\infty(\mathbb{R})$  and  $\text{supp } \hat{\psi}_1 \subset [-1, -\frac{1}{4}] \cup [\frac{1}{4}, 1]$ , and  $\psi_2 \in L^2(\mathbb{R})$  is a bump function satisfying  $\hat{\psi}_2 \in C^\infty(\mathbb{R})$ ,  $\text{supp } \hat{\psi}_2 \subset [-1, 1]$ , and  $\sum_{k \in \mathbb{Z}} |\hat{\psi}_2(k + \omega)|^2 = 1$  for  $\omega \in \mathbb{R}$  (cf. [19]). The associated *Shearlet Transform*  $\mathcal{SH}_\psi$  is then defined on  $L^2(\mathbb{R}^2)$  by

$$\mathcal{SH}_\psi f(j, k, m) = \langle f, \psi_{jkm} \rangle.$$

In order to provide an equal treatment of the direction of the  $x$ - and  $y$ -axis, the frequency plane is split into the cone

$$C = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1| \geq \frac{1}{4}, |\frac{\xi_2}{\xi_1}| \leq 1\},$$

its by  $90^\circ$  rotated copy, and the square centered at the origin of side length  $\frac{1}{2}$ . The Shearlet Transform acts on  $C$  and its copy as described above, while the choice of  $\psi$  has to be adapted appropriately. The center square can be filled in such a way that this system also forms a Parseval frame. The shearlet system in  $C$  and its copy is usually referred to as *shearlets on the cone*, see [19]. The associated tiling of the frequency plane is illustrated in Figure 1.

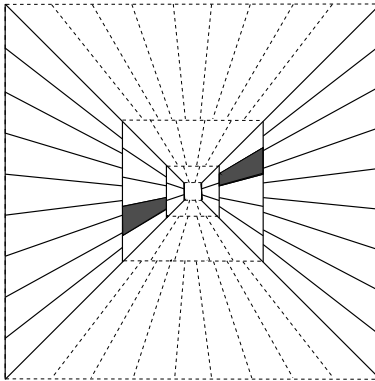


FIGURE 1. The tiling of the frequency domain induced by the shearlets on the cone.

The refinement matrices interesting to us for deriving a directional refinement of the lattice  $\mathbb{Z}^2$  are the dilation matrices used in (2.1) for  $j = 1$ , i.e., the matrices

$$M_k := S_{-k}A_{\frac{1}{4}} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2}k \\ 0 & \frac{1}{2} \end{pmatrix}, \quad k \in \mathbb{Z}.$$

Following the philosophy of the shearlets on the cone, also the matrices

$$\widetilde{M}_k := \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2}k & \frac{1}{4} \end{pmatrix},$$

which serve as dilation matrices for the rotated copy of  $C$ , will be employed as refinement matrices. The matrices  $M_k$  and  $\widetilde{M}_k$  not only provide the possibility to map a line to various directions, but moreover possess the property of refining the lattice  $\mathbb{Z}^2$  equally at each level as it is shown in the following result.

**Proposition 2.1.** *The following conditions hold.*

(i) *For all  $j, k \in \mathbb{Z}$ , we have*

$$M_k \in GL_2(\mathbb{R}) \quad \text{and} \quad M_k(4^{-j}\mathbb{Z} \times 2^{-j}\mathbb{Z}) = 4^{-(j+1)}\mathbb{Z} \times 2^{-(j+1)}\mathbb{Z}.$$

(ii) *For all  $j, k \in \mathbb{Z}$ , we have*

$$\widetilde{M}_k \in GL_2(\mathbb{R}) \quad \text{and} \quad \widetilde{M}_k(2^{-j}\mathbb{Z} \times 4^{-j}\mathbb{Z}) = 2^{-(j+1)}\mathbb{Z} \times 4^{-(j+1)}\mathbb{Z}.$$

*Proof.* (i) The first claim is obvious. To prove the second claim, let  $j, k \in \mathbb{Z}$  and  $m = (m_1, m_2) \in \mathbb{Z}^2$ . Then

$$M_k \begin{pmatrix} 4^{-j}m_1 \\ 2^{-j}m_2 \end{pmatrix} = \begin{pmatrix} 4^{-(j+1)}m_1 + 2^{-(j+1)}km_2 \\ 2^{-(j+1)}m_2 \end{pmatrix} = \begin{pmatrix} 4^{-(j+1)}(m_1 + 2^{j+1}km_2) \\ 2^{-(j+1)}m_2 \end{pmatrix},$$

which implies  $M_k(4^{-j}\mathbb{Z} \times 2^{-j}\mathbb{Z}) \subseteq 4^{-(j+1)}\mathbb{Z} \times 2^{-(j+1)}\mathbb{Z}$ .

Now let  $n = (n_1, n_2) \in \mathbb{Z}^2$ . Then choosing  $m = (m_1, m_2) \in \mathbb{Z}^2$  as  $m_1 = n_1 - 2^{j+1}kn_2$  and  $m_2 = n_2$  yields

$$M_k \begin{pmatrix} 4^{-j}m_1 \\ 2^{-j}m_2 \end{pmatrix} = \begin{pmatrix} 4^{-(j+1)}(n_1 - 2^{j+1}kn_2 + 2^{j+1}kn_2) \\ 2^{-(j+1)}n_2 \end{pmatrix} = \begin{pmatrix} 4^{-(j+1)}n_1 \\ 2^{-(j+1)}n_2 \end{pmatrix}.$$

Thus  $M_k(4^{-j}\mathbb{Z} \times 2^{-j}\mathbb{Z}) \supseteq 4^{-(j+1)}\mathbb{Z} \times 2^{-(j+1)}\mathbb{Z}$ , which proves the claim.

(ii) This follows by using similar arguments as in part (i).  $\square$

Thus, when applying a sequence of matrices  $M_{k_1}, \dots, M_{k_n}$  iteratively to the lattice  $\mathbb{Z}^2$ , at the  $j$ th level the points  $\{(4^{-j}(m_1 + \ell\frac{1}{4}), 2^{-j}(m_2 + \frac{1}{2})) : \ell \in \{1, 2, 3\}\}$  are added to the lattice  $4^{-j}\mathbb{Z} \times 2^{-j}\mathbb{Z}$ . This is true for an arbitrary choice of integers  $k_j \in \mathbb{Z}$ ,  $1 \leq j \leq n$ . Moreover, at each level this map is bijective.

A similar result holds for the matrices  $\widetilde{M}_k$ ,  $k \in \mathbb{Z}$ .

**2.2. Feasible Directions.** Let us now delve deeper into the explicit construction of the refinement by using the splitting idea of the shearlets on the cone. The overall aim is to provide a way of refinement such that the points on the  $y$ -axis – or any other line through the origin – can be moved to an arbitrary line through the origin during the refinement process. This immediately forces the refinement scheme to provide different strategies for refinement. We will see how this is can be achieved by using the matrices  $M_\varepsilon$  and  $\widetilde{M}_\varepsilon$  even only for  $\varepsilon = -1, 0, 1$ . In the sequel we will only focus on the matrices  $M_\varepsilon$ ,  $\varepsilon = -1, 0, 1$ , since the others can be treated simultaneously.

In the very first step of the refinement, we apply  $M_\varepsilon$  to  $\mathbb{Z}^2$  for  $\varepsilon = -1, 0, 1$ . Application of  $\varepsilon = 0$  does not change any directions,  $\varepsilon = 1$  maps the  $y$ -axis to the angle bisector in the first and third quadrant of the plane, and  $\varepsilon = -1$  has the same effect on the second and fourth quadrant. From now on, we consider the two cases  $\varepsilon \in \{0, -1\}$  or  $\varepsilon \in \{0, 1\}$  separately. Focusing on the second case, in each step we not only derive the refinement from a coarser scale  $4^{-j}\mathbb{Z} \times 2^{-j}\mathbb{Z}$  to a finer scale  $4^{-(j+1)}\mathbb{Z} \times 2^{-(j+1)}\mathbb{Z}$ , but also have two different ways to achieve this, either by applying  $M_0$  or by applying  $M_1$ . Hence, at the  $n$ th level we have applied a product of the form  $M_{\varepsilon_n} \dots M_{\varepsilon_1}$  to  $\mathbb{Z}^2$ , where  $\varepsilon_j \in \{0, 1\}$  for each  $1 \leq j \leq n$ . For  $\varepsilon \in \{0, -1\}$ , one can proceed in exactly the same way which we will, however, not work out in detail in this paper.

From now on, we will use the abbreviation  $E_n = \{0, 1\}^n$ ,  $n \in \mathbb{N}$ , for the index sets and will also denote by

$$E = \bigcup_{n \in \mathbb{N}} E_n$$

the set of all finite 0-1-sequences and by  $E_\infty = \{0, 1\}^\mathbb{N}$  the space of all infinite sequences. Note that  $E$  is canonically embedded in  $E_\infty$  by the mapping

$$E \ni \varepsilon \mapsto \varepsilon^* = (\varepsilon, 0, 0, \dots) \in E_\infty.$$

The main question to ask at this point concerns the possible directions this procedure allows us to map the points on the  $y$ -axis to. For this analysis, we restrict our attention to the first quadrant of the plane, since the same refinements occur in the third quadrant only in an origin-symmetric way.

We first notice that the sequence of  $n$  matrices  $M_\varepsilon$  we choose is completely determined by the associated sequence  $\varepsilon \in E_n$ . Hence this refinement scheme has the structure of a binary tree as illustrated in Figure 2.

The directions which might be obtained employing this refinement scheme are encoded in this binary tree in a special though natural way. To explore this relation, we first compute the product of the matrices which is applied to achieve the refinement at level  $n$ . Interestingly, the following binary number appears therein.

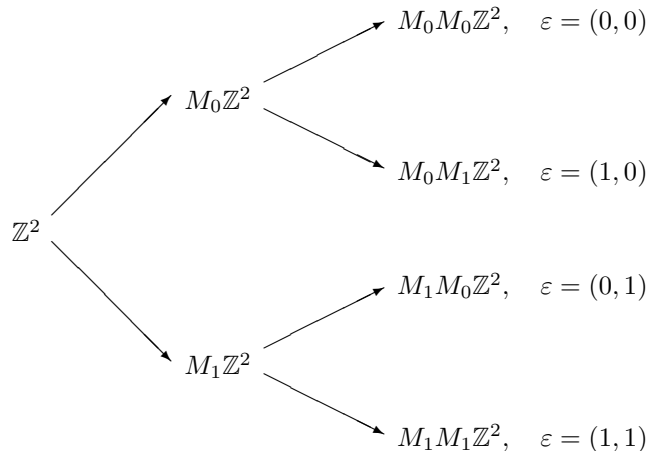


FIGURE 2. The binary tree up to level 2 associated with the refinement scheme.

**Notation 2.2.** For  $\varepsilon \in E_n$ ,  $n \in \mathbb{N}$ , we define

$$(\varepsilon)_2 = \sum_{j=0}^{n-1} \varepsilon_{j+1} 2^j \quad \text{and} \quad M_\varepsilon = M_{\varepsilon_n} \cdots M_{\varepsilon_1}.$$

Using this notion we obtain the following form for a refinement matrix  $M_\varepsilon$ .

**Lemma 2.3.** *Let  $n \in \mathbb{N}$  and  $\varepsilon \in E_n$ . Then we have*

$$M_\varepsilon = \begin{pmatrix} 4^{-n} & 4^{-n} 2 (\varepsilon)_2 \\ 0 & 2^{-n} \end{pmatrix}.$$

*Proof.* We will prove this lemma by induction. For  $n = 1$ , the claim obviously holds. Now suppose that the claim is true for some  $n \in \mathbb{N}$ . Let  $\varepsilon = (\varepsilon', \varepsilon_{n+1}) \in E_{n+1}$ ,  $\varepsilon' \in E_n$ . We have to distinguish between  $\varepsilon_{n+1} = 0$ , hence  $(\varepsilon)_2 = (\varepsilon')_2$ , with

$$M_\varepsilon = M_{(\varepsilon', \varepsilon_{n+1})} = \begin{pmatrix} 4^{-1} & 0 \\ 0 & 2^{-1} \end{pmatrix} \begin{pmatrix} 4^{-n} & 4^{-n} 2 (\varepsilon')_2 \\ 0 & 2^{-n} \end{pmatrix} = \begin{pmatrix} 4^{-(n+1)} & \frac{1}{2} 4^{-n} (\varepsilon')_2 \\ 0 & 2^{-(n+1)} \end{pmatrix},$$

and  $\varepsilon_{n+1} = 1$ , i.e.,  $(\varepsilon)_2 = (\varepsilon')_2 + 2^n$ , where

$$\begin{aligned} M_\varepsilon &= \begin{pmatrix} 4^{-1} & 2^{-1} \\ 0 & 2^{-1} \end{pmatrix} \begin{pmatrix} 4^{-n} & 4^{-n} 2 (\varepsilon')_2 \\ 0 & 2^{-n} \end{pmatrix} = \begin{pmatrix} 4^{-(n+1)} & \frac{1}{2} (4^{-n} (\varepsilon')_2 + 2^{-n}) \\ 0 & 2^{-(n+1)} \end{pmatrix} \\ &= \begin{pmatrix} 4^{-(n+1)} & \frac{1}{2} 4^{-n} ((\varepsilon')_2 + 2^n) \\ 0 & 2^{-(n+1)} \end{pmatrix} = \begin{pmatrix} 4^{-(n+1)} & \frac{1}{2} 4^{-n} (\varepsilon)_2 \\ 0 & 2^{-(n+1)} \end{pmatrix}. \end{aligned}$$

which advances the induction hypothesis.  $\square$

**Notation 2.4.** Let  $L$  be a line through the origin and  $\varepsilon \in E_n$ ,  $n \in \mathbb{N}$ . Then  $s(L, \varepsilon)$  denotes the slope of  $M_\varepsilon L$ , which is again a line through the origin. We further write  $s(L)$  for the slope of  $L$ .

The next result computes the values of the slopes  $s(L, \varepsilon)$ .

**Lemma 2.5.** *Let  $L$  be a line through the origin and  $\varepsilon \in E_n$ ,  $n \in \mathbb{N}$ . Then the following relations between  $s(L, \varepsilon)$ ,  $\varepsilon$  and the original  $L$  hold.*

(i) *If  $L$  is a line through the origin with  $s(L) \in (0, \infty)$ , then*

$$s(L, \varepsilon) = \frac{2^n}{\frac{1}{s(L)} + 2(\varepsilon)_2}.$$

(ii) *If  $L = \{0\} \times \mathbb{R}$ , i.e.,  $s(L) = \infty$ , then*

$$s(L, \varepsilon) = \frac{2^{n-1}}{(\varepsilon)_2},$$

*where we set  $2^{n-1}/0 := \infty$ .*

(iii) *If  $L = \mathbb{R} \times \{0\}$ , i.e.,  $s(L) = 0$ , then*

$$s(L, \varepsilon) = 0.$$

*Proof.* (i) We consider the point  $(1, s(L)) \in L$ . Using Lemma 2.3, we compute

$$M_\varepsilon \begin{pmatrix} 1 \\ s(L) \end{pmatrix} = \begin{pmatrix} 4^{-n} & 4^{-n} 2(\varepsilon)_2 \\ 0 & 2^{-n} \end{pmatrix} \begin{pmatrix} 1 \\ s(L) \end{pmatrix} = \begin{pmatrix} 4^{-n}(1 + 2s(L)(\varepsilon)_2) \\ 2^{-n}s(L) \end{pmatrix}$$

Hence, the slope of the line  $M_\varepsilon L$  equals

$$\frac{4^n s(L)}{2^n(1 + 2s(L)(\varepsilon)_2)} = \frac{2^n s(L)}{1 + 2s(L)(\varepsilon)_2} = \frac{2^n}{\frac{1}{s(L)} + 2(\varepsilon)_2}.$$

(ii) Here we consider the point  $(0, 1) \in L = \{0\} \times \mathbb{R}$ . Again employing Lemma 2.3, we obtain

$$M_\varepsilon \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4^{-n} & 4^{-n} 2(\varepsilon)_2 \\ 0 & 2^{-n} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4^{-n} 2(\varepsilon)_2 \\ 2^{-n} \end{pmatrix}.$$

Thus

$$s(L, \varepsilon) = \frac{4^n}{2^{n+1}(\varepsilon)_2} = \frac{2^n}{2(\varepsilon)_2}.$$

(iii) is easily verified by noting that the point  $(1, 0)$  is mapped to

$$M_\varepsilon \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4^{-n} & 4^{-n} 2(\varepsilon)_2 \\ 0 & 2^{-n} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4^{-n} \\ 0 \end{pmatrix}$$

so that the slope remains zero.  $\square$

Our main result in this section will show that indeed the points on an arbitrary line through the origin of slope  $\neq 0$  can be moved arbitrarily close to prescribed lines through the origin during the refinement process.

**Theorem 2.6.** *Let  $L$  be a line through the origin with  $s(L) \in (0, \infty]$ . Then, for each  $t \in [\frac{1}{2}, \infty]$  and  $\delta > 0$ , there exists some  $n \in \mathbb{N}$  and  $\varepsilon \in E_n$  such that*

$$|s(L, \varepsilon) - t| < \delta.$$

*Proof.* Suppose  $L$  is a line through the origin with  $s(L) \in (0, \infty)$ . The case  $s(L) = \infty$  can be dealt with in a similar way.

For given  $t \in (\frac{1}{2}, \infty)$  and  $\delta > 0$ , due to the denseness of rational numbers there exists some  $n \in \mathbb{N}$  and  $\varepsilon \in E_n$  such that

$$\left| \sum_{j=0}^{n-1} \varepsilon_{j+1} 2^{j-n+1} - \frac{1}{t} \right| < \frac{\delta}{t(t+\delta)} =: \tilde{\delta}.$$

Indeed,  $\varepsilon$  can be chosen as a truncation of the binary expansion of  $1/t$ . Note that without loss of generality we can assume that

$$\frac{1}{2^n s(L)} < \tilde{\delta},$$

since we can always enlarge  $n$ . Using these relations, we obtain

$$\begin{aligned} |s(L, \varepsilon) - t| &= \left| \frac{2^n}{\frac{1}{s(L)} + 2(\varepsilon)_2} - t \right| \\ &= \left| \frac{1}{\frac{1}{2^n s(L)} + \sum_{j=0}^{n-1} \varepsilon_{j+1} 2^{j-n+1}} - t \right| = \left| \frac{1 - t(\frac{1}{2^n s(L)} + \sum_{j=0}^{n-1} \varepsilon_{j+1} 2^{j-n+1})}{\frac{1}{2^n s(L)} + \sum_{j=0}^{n-1} \varepsilon_{j+1} 2^{j-n+1}} \right| \\ &\leq t \left| \frac{\frac{1}{t} - \sum_{j=0}^{n-1} \varepsilon_{j+1} 2^{j-n+1} - \frac{1}{2^n s(L)}}{\frac{1}{t} - \tilde{\delta}} \right| \leq \left| \frac{t^2 \tilde{\delta}}{1 - t \tilde{\delta}} \right| = \delta. \end{aligned}$$

Note that for the last equality we used  $\tilde{\delta} < \frac{1}{t}$ .

Now let  $\varepsilon \in E_\infty$  be defined by  $\varepsilon_j = 0$  for all  $j \geq j_0$  for some  $j_0 \in \mathbb{N}$ , and let  $M > 0$ . Then there exists some  $n \in \mathbb{N}$  such that

$$\frac{1}{2^n s(L)} + \sum_{j=0}^{n-1} \varepsilon_{j+1} 2^{j-n+1} < \frac{1}{M} \quad \text{for all } n \geq n_0,$$

which implies

$$s(L, (\varepsilon_1 \dots \varepsilon_n)) = \frac{1}{\frac{1}{2^n s(L)} + \sum_{j=0}^{n-1} \varepsilon_{j+1} 2^{j-n+1}} > M,$$

hence  $\lim_{n \rightarrow \infty} s(L, (\varepsilon_1 \dots \varepsilon_n)) = \infty$ .

Finally, let  $\varepsilon \in E_\infty$  be defined by  $\varepsilon_j = 1$  for all  $j \geq j_0$  for some  $j_0 \in \mathbb{N}$ . Then, for all  $n \in \mathbb{N}$ ,

$$s(L, (\varepsilon_1 \dots \varepsilon_n)) = \frac{2^n}{\frac{1}{s(L)} + \sum_{j=1}^n \varepsilon_j 2^j} = \frac{2^n}{\frac{1}{s(L)} + 2^{n+1} - 2 - \sum_{j=1}^{j_0-1} (1 - \varepsilon_j) 2^j}$$

and hence,

$$\lim_{n \rightarrow \infty} s(L, (\varepsilon_1 \dots \varepsilon_n)) = \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{2^n s(L)} + 2 - \frac{1}{2^{n-1}} - \frac{1}{2^n} \sum_{j=1}^{j_0-1} (1 - \varepsilon_j) 2^j} = \frac{1}{2}. \quad \square$$

Thus only employing  $M_0$  and  $M_1$  we can move any line arbitrarily close to any line of slope  $\in [\frac{1}{2}, \infty]$ . This shows the range of directions we might attain (compare Figure 3). However, we would like to mention that the change of orientation of the data induced by the subdivision scheme (see Definition 3.2) is also affected by directionality of the masks.

**Theorem 2.7.** *Let  $L$  be a line through the origin with  $s \in (0, \infty]$ . Then, for each  $t \in [-\frac{1}{2}, -\infty]$  and  $\delta > 0$ , there exists some  $n \in \mathbb{N}$  and  $\varepsilon \in E_n$  such that*

$$|s(L, \varepsilon) - t| < \delta.$$

Similar results as Theorems 2.6 and 2.7 also hold for the matrices  $\widetilde{M}_\varepsilon$ ,  $\varepsilon \in \{-1, 0, 1\}$ . We omit to also state these results for the sake of brevity, since they are similar to the previous theorems.

**2.3. A Directional Refinement of the Lattice  $\mathbb{Z}^2$ .** The results in the preceding section point out how to refine  $\mathbb{Z}^2$  in a directional way such that all possible directions can be attained. Dependent on whether we intend to map say the  $y$ -axis to a line with a slope contained in  $[\frac{1}{2}, \infty]$ ,  $[-\frac{1}{2}, -\infty]$ , or  $[-\frac{1}{2}, \frac{1}{2}]$ , we choose to refine by using the matrices  $M_0, M_1, M_{-1}, \widetilde{M}_0, \widetilde{M}_1$ , respectively. Once the type of matrices is chosen, we iterate depending on the angle we would like to attain by using Theorem 2.6, Theorem 2.7, or the corresponding result for the matrices  $\widetilde{M}_\varepsilon$ ,  $\varepsilon \in \{-1, 0, 1\}$ . For an illustration of the different areas of lines through the origin which can be attained during the refinement process dependent on the chosen matrices we refer to Figure 3.

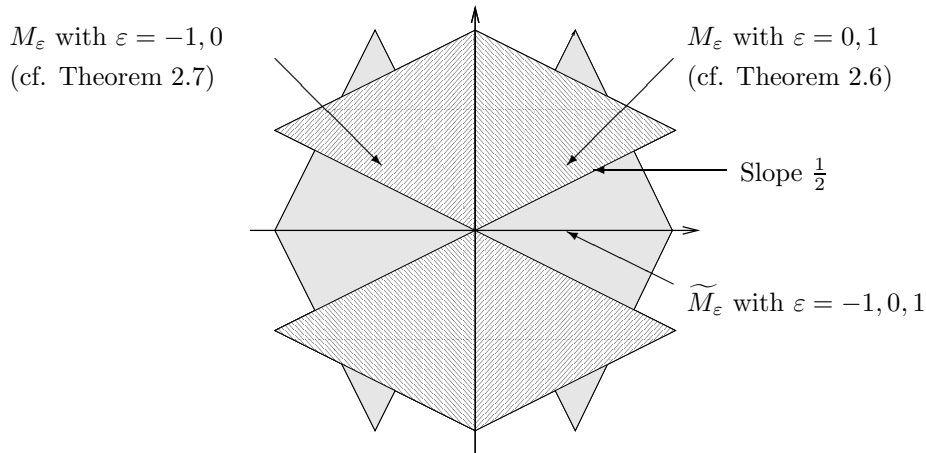


FIGURE 3. This figure shows the different areas of lines through the origin which can be attained during the refinement process depending on the choice of  $M_\varepsilon$  and  $\widetilde{M}_\varepsilon$  and  $\varepsilon \in \{-1, 0, 1\}$ .

From now on we will focus entirely on the matrices  $M_0$  and  $M_1$ . All following results can be derived in a similar way for  $M_{-1}, M_0$  and for  $\widetilde{M}_{-1}, \widetilde{M}_0, \widetilde{M}_1$ .

### 3. ADAPTIVE DIRECTIONAL SUBDIVISION

In this section, we finally arrive at the announced definition of a new type of subdivision schemes, based on the interaction of *two* “normal” stationary subdivision schemes, which we will study in the sequel. To that end, we choose two *masks*  $a_\varepsilon$ ,  $\varepsilon \in \{0, 1\}$ , i.e., *finitely supported* sequences  $a_\varepsilon \in \ell_{00}(\mathbb{Z}^2)$  as well as the expanding scaling matrices  $W_\varepsilon = M_\varepsilon^{-1}$ ,  $\varepsilon \in \{0, 1\}$ . These matrices can be given explicitly as

$$(3.1) \quad W_0 = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad W_1 = \begin{pmatrix} 4 & -4 \\ 0 & 2 \end{pmatrix},$$

and again we set  $W_\varepsilon = W_{\varepsilon_n} \cdots W_{\varepsilon_1}$ ,  $\varepsilon \in E_n$ . Also note that

$$W_1 = W_0 \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} W_0.$$

Such a decomposition also exists for the iterated matrices  $W_\varepsilon$ ,  $\varepsilon \in \{0, 1\}^n$ ,  $n \in \mathbb{N}$ .

To formulate the next auxiliary result, we also define for  $\varepsilon \in E_n$  the dyadic number

$$[\varepsilon]_2 = .\varepsilon_1 \dots \varepsilon_n := \sum_{j=1}^n \varepsilon_j 2^{-j} \in [0, 1].$$

With this notation at hand, we obtain the following counterpiece of Lemma 2.3.

**Lemma 3.1.** *For  $n \in \mathbb{N}_0$  and  $\varepsilon \in E_n$ , we have*

$$W_\varepsilon = W_{\varepsilon_n} \cdots W_{\varepsilon_1} = \begin{pmatrix} 4^n & -4^n 2 [\varepsilon]_2 \\ 0 & 2^n \end{pmatrix} = U_\varepsilon W_0^n = W_0^n V_\varepsilon,$$

where

$$U_\varepsilon = \begin{pmatrix} 1 & -2^{n+1} [\varepsilon]_2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad V_\varepsilon = \begin{pmatrix} 1 & -2 [\varepsilon]_2 \\ 0 & 1 \end{pmatrix},$$

hence  $U_\varepsilon = V_\varepsilon^{2^n}$ .

*Proof.* The proof is again of inductive nature and relies on noting that

$$W_0 W_\varepsilon = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 4^n & -4^n 2 [\varepsilon]_2 \\ 0 & 2^{n+1} \end{pmatrix} = \begin{pmatrix} 4^{n+1} & -4^{n+1} 2 [(\varepsilon, 0)]_2 \\ 0 & 2^{n+2} \end{pmatrix}$$

as well as

$$\begin{aligned} W_1 W_\varepsilon &= \begin{pmatrix} 4 & -4 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 4^n & -4^n 2 [\varepsilon]_2 \\ 0 & 2^n \end{pmatrix} = \begin{pmatrix} 4^{n+1} & -4^{n+1} (2 [\varepsilon]_2 + 2^{-n}) \\ 0 & 2^{n+1} \end{pmatrix} \\ &= \begin{pmatrix} 4^{n+1} & -4^{n+1} 2 [(\varepsilon, 1)]_2 \\ 0 & 2^{n+1} \end{pmatrix}. \end{aligned}$$

Hence,

$$W_\varepsilon = \begin{pmatrix} 2^{2n} & -2^{2n+1} [\varepsilon]_2 \\ 0 & 2^n \end{pmatrix} = W_0^n \begin{pmatrix} 1 & -2 [\varepsilon]_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2^{n+1} [\varepsilon]_2 \\ 0 & 1 \end{pmatrix} W_0.$$

Since for  $x \in \mathbb{R}$

$$\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & -kx \\ 0 & 1 \end{pmatrix},$$

also the final claim follows.  $\square$

Note that  $V_{(0, \dots, 0)}$ ,  $V_{(1, 0, \dots, 0)}$ , and all  $U_\varepsilon$  are *unimodular* matrices, i.e., they have an inverse in  $\mathbb{Z}^{2 \times 2}$ . A particular role will be played by the two matrices

$$V = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad U = V^2 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

which satisfy

(3.2)

$$W_1 = U W_0 = W_0 V, \quad \text{i.e.} \quad W_1 = U^{-1} W_1 V \quad \text{and} \quad W_0 = U W_0 V^{-1}.$$

The associated subdivision schemes are now defined as follows. The term *adaptive* refers to the tree-like structure, which provides various branches for subdivision, whereas the term *directional* refers to the directional structure which comes from the shearing process contained in the dilation matrices  $W_\varepsilon$ ,  $\varepsilon \in E$ .

**Definition 3.2.** Let  $a_\varepsilon \in \ell_{00}(\mathbb{Z}^2)$ ,  $\varepsilon \in \{0, 1\}$  be two masks, that is, two *finitely supported sequences*, and let  $W_\varepsilon$ ,  $\varepsilon \in \{0, 1\}$  be defined as in (3.1). Then the associated *adaptive directional subdivision scheme of order  $n$*  is defined by

$$S_\varepsilon = S_{\varepsilon_n} \cdots S_{\varepsilon_1}, \quad \varepsilon \in E_n, \quad n \in \mathbb{N},$$

where, for  $\eta \in \{0, 1\}$ ,

$$S_\eta c := S_{a_\eta, W_\eta} c := \sum_{\alpha \in \mathbb{Z}^2} a_\eta(\cdot - W_\eta \alpha) c(\alpha), \quad c \in \ell_\infty(\mathbb{Z}^2),$$

Note that both the mask as well as the scaling matrix of these subdivision schemes depend on the index  $\varepsilon$ . Moreover, we wish to remark that these schemes can clearly be computed in a tree-like fashion by setting

$$S_\varepsilon c = S_{(\varepsilon', \varepsilon_n)} c = S_{\varepsilon_n} S_{\varepsilon'} c = \sum_{\beta \in \mathbb{Z}^2} a_{\varepsilon_n}(\cdot - W_{\varepsilon_n} \beta) S_{\varepsilon'} c(\beta), \quad \varepsilon' \in E_{n-1}.$$

Adaptive directional subdivision schemes can be considered subdivision schemes of their own, however, with a different scaling matrix. This is easily seen by means of the following example: for  $\alpha \in \mathbb{Z}^2$  we have

$$\begin{aligned} S_{(\varepsilon_1, \varepsilon_2)} c &= S_{\varepsilon_2} S_{\varepsilon_1} c = \sum_{\beta \in \mathbb{Z}^2} a_{\varepsilon_2}(\cdot - W_{\varepsilon_2} \beta) (S_{\varepsilon_1} c)(\beta) \\ &= \sum_{\beta \in \mathbb{Z}^2} a_{\varepsilon_2}(\cdot - W_{\varepsilon_2} \beta) \sum_{\gamma \in \mathbb{Z}^2} a_{\varepsilon_1}(\beta - W_{\varepsilon_1} \gamma) c(\gamma) \\ &= \sum_{\gamma \in \mathbb{Z}^2} \left[ \sum_{\beta \in \mathbb{Z}^2} a_{\varepsilon_2}(\cdot - W_{\varepsilon_2} \beta - W_{\varepsilon_2} W_{\varepsilon_1} \gamma) a_{\varepsilon_1}(\beta) \right] c(\gamma) \\ &=: \sum_{\gamma \in \mathbb{Z}^2} a_{(\varepsilon_1, \varepsilon_2)}(\cdot - W_{(\varepsilon_1, \varepsilon_2)} \gamma) c(\gamma). \end{aligned}$$

An inductive application of this argument immediately gives the next result.

**Lemma 3.3.** For  $\varepsilon \in E_n$ , the subdivision scheme  $S_\varepsilon$  acts as

$$S_\varepsilon c(\alpha) = \sum_{\beta \in \mathbb{Z}^2} a_\varepsilon(\alpha - W_\varepsilon \beta) c(\beta), \quad \alpha \in \mathbb{Z}^2,$$

where the coefficient sequences  $a_\varepsilon$  are recursively defined as  $a_\varepsilon = a_{(\varepsilon', \varepsilon_n)} = S_{\varepsilon_n} a_{\varepsilon'}$ .

To get a better understanding of the geometry of adaptive directional subdivision, we write  $a_1$  as  $a_1 = \tilde{a}_0(U \cdot)$  which is always possible since  $U$  is unimodular. It then follows from repeated applications of (3.2) that

$$\begin{aligned} S_{a_1, W_1} c &= \sum_{\alpha \in \mathbb{Z}^2} a_1(\cdot - W_1 \alpha) c(\alpha) \\ &= \sum_{\alpha \in \mathbb{Z}^2} \tilde{a}_0(U \cdot - U W_1 U^{-1} U \alpha) c(\alpha) \\ &= \sum_{\alpha \in \mathbb{Z}^2} \tilde{a}_0(U \cdot - U W_1 V^{-2} \alpha) c(U^{-1} \alpha) \\ &= \sum_{\alpha \in \mathbb{Z}^2} \tilde{a}_0(U \cdot - U W_0 V^{-1} \alpha) c(U^{-1} \alpha) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\alpha \in \mathbb{Z}^2} \tilde{a}_0(U \cdot -W_0\alpha) c(U^{-1}\alpha) \\
&= (S_{\tilde{a}_0, W_0} c(U^{-1}\cdot))(U\cdot).
\end{aligned}$$

This identity can be rewritten in terms of dilation operators as

$$S_1 = D_U \tilde{S}_0 D_{U^{-1}} = D_U \tilde{S}_0 D_U^{-1}, \text{ hence } S_{(1, \dots, 1)} = D_U \tilde{S}_{(0, \dots, 0)} D_U^{-1},$$

and enables us to implement the subdivision scheme  $S_1$  in terms of  $\tilde{S}_0$  and the shear operator  $D_U$ . Moreover, it explains the geometry of the scheme  $S_1$ : first, a shearing by  $U^{-1}$  is applied to the data sequence, then the subdivision operator refines the data in the sheared direction with a higher resolution than the data in the non-sheared direction, so that the additional application of the shearing by  $U$  does not fully compensate the initial one. In summary, this process leads to limit functions which are sheared versions of the limit function of  $S_0$  and the amount of shearing is determined by when and how often  $S_1$  is applied in the process. We remark that this geometry is very much in the spirit of the Continuous Shearlet Transform, which can be regarded as applying a shearing operator, an anisotropic 2-D Wavelet Transform, and again a shearing operator [23].

#### 4. CONVERGENCE

In this section, we shall study convergence of the previously introduced adaptive directional subdivision schemes. To that end, we introduce the *projection operators*  $P_n : E_\infty \rightarrow E_n$ ,  $n \in \mathbb{N}$ , which extract the initial segment of order  $n$  from a sequence:  $P_n \varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ .

**Definition 4.1.** The adaptive directional subdivision scheme is said to be *convergent in  $C(\mathbb{R}^2)$* , if, for any  $\varepsilon \in E_\infty$ , there exists a nonzero uniformly continuous function  $f_\varepsilon \in C(\mathbb{R}^2)$  such that

$$\lim_{n \rightarrow \infty} \sup_{\alpha \in \mathbb{Z}^2} |f_\varepsilon(W_{P_n \varepsilon}^{-1} \alpha) - S_{P_n \varepsilon} \delta(\alpha)| = 0.$$

Note that this is equivalent to

$$\lim_{n \rightarrow \infty} \sup_{\alpha \in \mathbb{Z}^2} |f_\varepsilon(W_{P_n \varepsilon}^{-1} \alpha) - a_{P_n \varepsilon}(\alpha)| = 0.$$

Since any sequence  $c \in \ell(\mathbb{Z}^2)$  can be trivially written as

$$c = \sum_{\alpha \in \mathbb{Z}^2} c(\alpha) \delta(\cdot - \alpha), \quad \delta(\alpha) := \delta_{\alpha, 0},$$

and since the subdivision operator is linear, we immediately obtain the following convolution style representation of the limit function.

**Proposition 4.2.** *If the adaptive directional subdivision scheme converges for some  $\varepsilon \in E_\infty$  then the limit function takes the form*

$$f_\varepsilon * c = \sum_{\alpha \in \mathbb{Z}^2} c(\alpha) f_\varepsilon(\cdot - \alpha).$$

**4.1. Basic Properties.** This definition of convergence has an immediate consequence: If the adaptive directional subdivision scheme is a convergent one, then, in particular,  $a_0$  and  $a_1$  must define convergent adaptive directional subdivision schemes, which follows by simply choosing  $\varepsilon = (0, 0, \dots)$  and  $\varepsilon = (1, 1, \dots)$ , respectively. Consequently, they must both preserve constants.

**Lemma 4.3.** *If the adaptive directional subdivision scheme is convergent, then*

$$(4.1) \quad \sum_{\beta \in \mathbb{Z}^2} a_\varepsilon(\alpha + W_\varepsilon \beta) = 1, \quad \alpha \in \mathbb{Z}^2, \quad \varepsilon \in \{0, 1\}.$$

An alternative but equivalent definition of convergence of a adaptive directional subdivision scheme can be given in terms of function spaces instead of sequence spaces by means of test functions.

**Definition 4.4.** A function  $g \in C(\mathbb{R}^2)$  is called a *test function*, if it is compactly supported and its integer translates form a stable partition of unity, that is,

- (i)  $\sum_{\alpha} g(\cdot - \alpha) = 1$ ,
- (ii) there exist constants  $0 < A < B < \infty$  such that for any  $c \in \ell_\infty$

$$A \|c\|_\infty \leq \|g * c\|_\infty \leq B \|c\|_\infty, \quad g * c := \sum_{\alpha \in \mathbb{Z}^2} c(\alpha) g(\cdot - \alpha).$$

The most prominent examples for test functions are the tensor product B-Splines so that there even exist *refinable* test functions of arbitrary regularity. With the help of test functions, convergence can be described as follows.

**Theorem 4.5.** *The adaptive directional subdivision scheme converges if and only if for any  $\varepsilon \in E_\infty$  there exists a nonzero uniformly continuous function  $f_\varepsilon$  such that*

$$(4.2) \quad \lim_{n \rightarrow \infty} \|f_\varepsilon - (g * S_{P_n, \varepsilon} \delta)(W_{P_n, \varepsilon})\|_\infty = 0$$

- (i) for some test function  $g$ .
- (ii) for any test function  $g$ .

*Proof.* For classical subdivision, this result is due to Dahmen and Micchelli [13] and we just show how it can be extended in a straightforward way to adaptive directional subdivision. To that end, let  $g$  be any test function and recall that for any *uniformly continuous* function  $f$  and any expanding matrix  $M$  the “quasi-interpolant”

$$g * \sigma_M f = \sum_{\alpha \in \mathbb{Z}^2} f(M\alpha) g(\cdot - \alpha),$$

with the *sampling operator*  $\sigma_M := (f(M\alpha) : \alpha \in \mathbb{Z}^2)$ , satisfies

$$\|f - g * \sigma_{M^{-1}} f(M\cdot)\|_\infty \leq C_g \omega(f, \|M^{-1}\|),$$

where

$$\omega(f, \delta) := \sup_{x \in \mathbb{R}^2} \sup_{\|x-y\|_\infty \leq \delta} |f(x) - f(y)|,$$

denotes the modulus of continuity of  $f$ . Recall that  $\omega(f, \delta) \rightarrow 0$  for  $\delta \rightarrow 0$  as long as  $f$  is uniformly continuous. Now, we have that

$$\begin{aligned} & \|f_\varepsilon - (g * S_{P_n \varepsilon} \delta)(W_{P_n \varepsilon} \cdot)\|_\infty \\ & \leq \left\| f_\varepsilon - \left( g * \sigma_{W_{P_n \varepsilon}^{-1}} f_\varepsilon \right) (W_{P_n \varepsilon} \cdot) \right\|_\infty + \left\| g * \left( \sigma_{W_{P_n \varepsilon}^{-1}} f_\varepsilon - S_{P_n \varepsilon} \delta \right) (W_{P_n \varepsilon} \cdot) \right\|_\infty \\ & = \left\| f_\varepsilon - \left( g * \sigma_{W_{P_n \varepsilon}^{-1}} f_\varepsilon \right) (W_{P_n \varepsilon} \cdot) \right\|_\infty + \left\| g * \left( \sigma_{W_{P_n \varepsilon}^{-1}} f_\varepsilon - S_{P_n \varepsilon} \delta \right) \right\|_\infty \\ & \leq C_g \omega(f, \|W_{P_n \varepsilon}^{-1}\|) + B \left\| \sigma_{W_{P_n \varepsilon}^{-1}} f_\varepsilon - S_{P_n \varepsilon} \delta \right\|_\infty. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \left\| \sigma_{W_{P_n \varepsilon}^{-1}} f_\varepsilon - S_{P_n \varepsilon} \delta \right\|_\infty \leq A^{-1} \left\| g * \left( \sigma_{W_{P_n \varepsilon}^{-1}} f_\varepsilon - S_{P_n \varepsilon} \delta \right) (W_{P_n \varepsilon} \cdot) \right\|_\infty \\ & \leq A^{-1} \left( \left\| f_\varepsilon - \left( g * \sigma_{W_{P_n \varepsilon}^{-1}} f_\varepsilon \right) (W_{P_n \varepsilon} \cdot) \right\|_\infty + \left\| f_\varepsilon - (g * S_{P_n \varepsilon} \delta)(W_{P_n \varepsilon} \cdot) \right\|_\infty \right) \\ & \leq A^{-1} (C_g \omega(f, \|W_{P_n \varepsilon}^{-1}\|) + \|f_\varepsilon - (g * S_{P_n \varepsilon} \delta)(W_{P_n \varepsilon} \cdot)\|_\infty) \end{aligned}$$

which verifies the equivalence. Since therefore convergence of the adaptive directional subdivision scheme is equivalent to (4.2) holding for an arbitrary test function, this property holds for one particular test function if and only if it holds for any test function.  $\square$

**Theorem 4.6.** *If the adaptive directional subdivision scheme converges, then the limit functions  $f_\varepsilon$ ,  $\varepsilon \in E_\infty$ , satisfy the refinement equation*

$$(4.3) \quad f_\varepsilon = \sum_{\alpha \in \mathbb{Z}^2} a_{\varepsilon_1}(\alpha) f_{\widehat{\varepsilon}}(W_{\varepsilon_1} \cdot - \alpha), \quad \widehat{\varepsilon} := (\varepsilon_2, \varepsilon_3, \dots).$$

*Proof.* We define the *transition operator*

$$T_\varepsilon f = \sum_{\alpha \in \mathbb{Z}^2} a_\varepsilon(\alpha) f(W_\varepsilon \cdot - \alpha), \quad f \in C(\mathbb{R}^2), \quad \varepsilon \in \{0, 1\}$$

and note that, for  $c \in \ell_\infty$ ,

$$\begin{aligned} (T_\varepsilon f) * c &= \sum_{\alpha \in \mathbb{Z}^2} T_\varepsilon f(\cdot - \alpha) c(\alpha) = \sum_{\alpha, \beta \in \mathbb{Z}^2} a_\varepsilon(\beta) c(\alpha) f(W_\varepsilon \cdot - W_\varepsilon \alpha - \beta) \\ &= \sum_{\beta \in \mathbb{Z}^2} \left( \sum_{\alpha \in \mathbb{Z}^2} a_\varepsilon(\beta - W_\varepsilon \alpha) c(\alpha) \right) f(W_\varepsilon \cdot - \beta) = (f * S_\varepsilon c)(W_\varepsilon \cdot). \end{aligned}$$

By iteration, we then find for  $\varepsilon \in \{0, 1\}^n$  that

$$\begin{aligned} (f * S_\varepsilon c)(W_\varepsilon \cdot) &= (f * S_{\varepsilon_n} \cdots S_{\varepsilon_1} c)(W_{\varepsilon_n} \cdots W_{\varepsilon_1} \cdot) \\ &= (T_{\varepsilon_n} f * S_{\varepsilon_{n-1}} \cdots S_{\varepsilon_1} c)(W_{\varepsilon_{n-1}} \cdots W_{\varepsilon_1} \cdot) = \dots = (T_\varepsilon f * c) \end{aligned}$$

where

$$T_\varepsilon f = T_{\varepsilon_1} \cdots T_{\varepsilon_n} f, \quad \varepsilon \in \{0, 1\}^n.$$

Since, for  $n \in \mathbb{N}$ ,

$$\begin{aligned} T_{\varepsilon_1} f_{\widehat{\varepsilon}} &= T_{\varepsilon_1} f_{\widehat{\varepsilon}} * \delta = (f_{\widehat{\varepsilon}} * S_{\varepsilon_1} \delta)(W_{\varepsilon_1} \cdot) \\ &= [(f_{\widehat{\varepsilon}} - (g * S_{P_{n-1} \widehat{\varepsilon}} \delta)(W_{P_{n-1} \widehat{\varepsilon}} \cdot)) * S_{\varepsilon_1} \delta](W_{\varepsilon_1} \cdot) \\ &\quad + [(g * S_{P_{n-1} \widehat{\varepsilon}} \delta)(W_{P_{n-1} \widehat{\varepsilon}} \cdot) * S_{\varepsilon_1} \delta](W_{\varepsilon_1} \cdot) \\ &= [(f_{\widehat{\varepsilon}} - (g * S_{P_{n-1} \widehat{\varepsilon}} \delta)(W_{P_{n-1} \widehat{\varepsilon}} \cdot)) * S_{\varepsilon_1} \delta](W_{\varepsilon_1} \cdot) + (g * S_{P_n \varepsilon} \delta)(W_{P_n \varepsilon} \cdot), \end{aligned}$$

it follows that

$$\begin{aligned} & \|T_{\varepsilon_1} f_{\widehat{\varepsilon}} - f_{\varepsilon}\|_{\infty} \\ & \leq \| (f_{\widehat{\varepsilon}} - (g * S_{P_{n-1}\widehat{\varepsilon}})(W_{P_{n-1}\widehat{\varepsilon}})) * S_{\varepsilon_1} \delta \|_{\infty} + \|f_{\varepsilon} - (g * S_{P_n \varepsilon} \delta)(W_{P_n \varepsilon})\|_{\infty} \\ & \leq \|S_{\varepsilon_1}\| \|f_{\widehat{\varepsilon}} - (g * S_{P_{n-1}\widehat{\varepsilon}})(W_{P_{n-1}\widehat{\varepsilon}})\|_{\infty} + \|f_{\varepsilon} - (g * S_{P_n \varepsilon} \delta)(W_{P_n \varepsilon})\|_{\infty} \end{aligned}$$

and the right hand side of this inequality converges to zero for  $n \rightarrow \infty$  while the left hand side is independent of  $n$ . Thus  $T_{\varepsilon_1} f_{\widehat{\varepsilon}} = f_{\varepsilon}$  which is (4.3).  $\square$

#### 4.2. An Algebraic Description, Sum Rules and Polynomial Reproduction.

Next, we give a more detailed description of the necessary condition (4.1) from Lemma 4.3 in algebraic terms. To that end, we recall the definition of the *symbol* of a mask  $a$ , defined as

$$a^*(z) = \sum_{\alpha \in \mathbb{Z}^2} a(\alpha) z^{\alpha}, \quad z \in \mathbb{C}_*^2 = (\mathbb{C} \setminus \{0\})^2,$$

as well as the *subsymbols*

$$a_{\varepsilon, \eta}^*(z) = \sum_{\alpha \in \mathbb{Z}^2} a(\eta + W_{\varepsilon} \alpha) z^{\alpha}, \quad \eta \in H_{\varepsilon} := W_{\varepsilon}^T [0, 1]^2 \cap \mathbb{Z}^2, \quad \varepsilon \in \{0, 1\}.$$

The symbol can be “reconstructed” from the subsymbols by the well-known formula

$$a^*(z) = \sum_{\eta \in H_{\varepsilon}} z^{\eta} a_{\varepsilon, \eta}^*(z^{W_{\varepsilon}}), \quad \varepsilon \in \{0, 1\},$$

from which the following result follows immediately, cf. [29].

**Proposition 4.7.** *The mask  $a_{\varepsilon}$  satisfies (4.1), the sum rule of order 0, if and only if*

$$a^*(z) = 0, \quad z \in \left\{ e^{-2\pi i W_{\varepsilon}^{-T} \eta} : \eta \in H_{\varepsilon} \setminus \{0\} \right\}.$$

For a more algebraic description, we need the notion of a *quotient ideal*. Recall that an ideal in  $\Lambda$ , the ring of *Laurent polynomials* in two variables, is a subset of  $\Lambda$  that is closed under addition and multiplication by arbitrary Laurent polynomials. The *quotient ideal* of two Laurent ideals  $I, J$ , is defined as

$$I : J := \{f \in \Lambda : f \cdot J \subseteq I\}$$

and has the almost obvious property that  $I \subseteq I : J$ . For any matrix  $X \in \mathbb{Z}^{2 \times 2}$ , with column vectors  $x_1, x_2$  we finally define the ideal

$$\langle z^X - 1 \rangle := \langle z^{x_1} - 1, z^{x_2} - 1 \rangle := \{f_1(z)(z^{x_1} - 1) + f_2(z)(z^{x_2} - 1) : f_1, f_2 \in \Lambda\}$$

and its special case  $\langle z - 1 \rangle := \langle z^I - 1 \rangle$ . Then we have the following result from [26].

**Theorem 4.8.** *The mask  $a_{\varepsilon}$  satisfies (4.1), the sum rule of order 0, if and only if*

$$a^* \in \langle z^{W_{\varepsilon}} - 1 \rangle : \langle z - 1 \rangle.$$

To conveniently formulate an important consequence of this theorem, we introduce the vectors

$$[z^X - 1] = \begin{bmatrix} z^{x_1} - 1 \\ z^{x_2} - 1 \end{bmatrix}, \quad X = [x_1, x_2] \in \mathbb{Z}^{2 \times 2}.$$

With this notation we have the following result.

**Corollary 4.9.** *If the adaptive directional subdivision scheme converges, then there exist matrix valued masks  $B_\varepsilon$ ,  $\varepsilon \in \{0, 1\}$  such that*

$$(4.4) \quad [z - 1] a_\varepsilon^*(z) = B_\varepsilon^*(z) [z^{W_\varepsilon} - 1], \quad \varepsilon \in \{0, 1\}.$$

*Proof.* Any convergent subdivision must satisfy the sum rule of order 0 for  $a_\varepsilon$ ,  $\varepsilon \in \{0, 1\}$ , and so, by Theorem 4.8, it follows for  $\varepsilon \in \{0, 1\}$  and  $j = 1, 2$  that

$$(z_j - 1) a_\varepsilon^*(z) = b_{j1}^*(z) (z^{(W_\varepsilon)_1} - 1) + b_{j2}^*(z) (z^{(W_\varepsilon)_2} - 1).$$

Written in matrix form, this is what has been claimed.  $\square$

**Definition 4.10.** The matrix masks  $B_\varepsilon$ ,  $\varepsilon \in \{0, 1\}$ , from (4.4) are called *representation masks* of  $a_\varepsilon$ ,  $\varepsilon \in \{0, 1\}$ , respectively.

*Remark 4.11.* Recall that the computation of the representation masks  $B_\varepsilon$  can be performed by *reduction*, a multivariate generalization of division with remainder, see [10, 27] for the term order and homogeneous versions of this process, respectively. Therefore, the symbolic determination of  $B_\varepsilon$  can easily be done with the help of practically any Computer Algebra system that supports constructive polynomial ideal theory.

Note however, that the representation masks are *not* unique to the appearance of *syzygies* of  $[z^{W_\varepsilon} - 1]$ , not even if an H-representation, cf. [25], is chosen where – in the case of  $W_0$  – we have the “minimal degree” requirements that

$$\deg b_{11} = \deg b_{21} = \deg a_0 - 3, \quad \deg b_{12} = \deg b_{22} = \deg a_0 - 1,$$

see also [28].

We continue by giving explicit bases of the quotient ideals for our specific choice of  $W_\varepsilon$ . This is easy for  $W_0$  as all entries in this matrix are nonnegative, and indeed it is not difficult to see that

$$\begin{aligned} I_0 &:= \langle z^{W_0} - 1 \rangle : \langle z - 1 \rangle = \langle z_1^4 - 1, z_2^2 - 1 \rangle : \langle z - 1 \rangle \\ &= \langle (z_1^3 + z_1^2 + z_1 + 1)(z_2 + 1) \rangle + \langle z_1^4 - 1 \rangle + \langle z_2^2 - 1 \rangle. \end{aligned}$$

In fact, the graded homogeneous leading terms of the above ideal basis are  $z_1^3 z_2$ ,  $z_1^4$  and  $z_2^2$  so that the quotient space is spanned exactly by the seven monomials

$$1, z_1, z_2, z_1^2, z_1 z_2, z_1^3, z_1^2 z_2,$$

and their number coincides with the number of joint zeros of  $I_0$ . Hence, by the same reasoning as in [26, 28] they even form a graded Gröbner basis, hence an H-basis of the ideal  $I_0$ . Recall that a subset  $H$  of an ideal  $I$  is called an *H-basis*, if any polynomial  $f \in I$  can be written in the form

$$f = \sum_{h \in H} f_h h, \quad \deg f \geq \deg f_h + \deg h,$$

where  $\deg$  denotes, as usual, the *total degree* of a polynomial. We will also use  $\Pi_n$  for the vector space of all polynomials of total degree at most  $n$ .

The situation for  $I_1 = \langle z^{W_1} - 1 \rangle$  appears to be a little bit more intricate due to the appearance of a negative entry in  $W_1$ . Here it is helpful to recall that  $W_1 = UW_0$ ,  $U = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ , to define  $y = z^U = (z_1, z_1^{-2} z_2)$ , hence also  $z = y^{U^{-1}} =$

$(y_1, y_1^2 y_2)$  and to realize that

$$I_1 = \langle z^{W_1} - 1 \rangle : \langle z - 1 \rangle = \langle z^{UW_0} - 1 \rangle : \langle z - 1 \rangle = \langle y^{W_0} - 1 \rangle : \langle y^{U^{-1}} - 1 \rangle$$

Since

$$\begin{aligned} \langle y^{U^{-1}} - 1 \rangle &= \langle y_1 - 1, y_1^2 y_2 - 1 \rangle = \langle y_1 - 1, y_1^2 y_2 - (y_1 y_2 + y_2)(y_1 - 1) - 1 \rangle \\ &= \langle y_1 - 1, y_2 - 1 \rangle = \langle y - 1 \rangle, \end{aligned}$$

we thus obtain that

$$\begin{aligned} I_1 &= \langle y^{W_0} - 1 \rangle : \langle y - 1 \rangle \\ &= \langle (y_1^3 + y_1^2 + y_1 + 1)(y_2 + 1) \rangle + \langle y_1^4 - 1 \rangle + \langle y_2^2 - 1 \rangle \\ &= \langle (z_1^3 + z_1^2 + z_1 + 1)(z_2 + z_1^2) \rangle + \langle z_1^4 - 1 \rangle + \langle z_2^2 - z_1^4 \rangle \end{aligned}$$

To arrive at the somewhat surprising observation that in fact  $I_1 = I_0$ , we add  $z_1^4 - 1$  to the third basis element,  $z_2^2 - z_1^4$ , yielding  $z_2^2 - 1$  again, and subtract  $(z_1 + 1)(z_1^4 - 1)$  from the first basis element which leads to

$$\begin{aligned} &(z_1^3 + z_1^2 + z_1 + 1)(z_2 + z_1^2) - (z_1 + 1)(z_1^4 - 1) \\ &= (z_1^3 + z_1^2 + z_1 + 1)z_2 + z_1^5 + z_1^4 + z_1^3 + z_1^2 - z_1^5 - z_1^4 + z_1 + 1 \\ &= (z_1^3 + z_1^2 + z_1 + 1)(z_2 + 1) \end{aligned}$$

and therefore to the following result.

**Theorem 4.12.** *The two quotient ideals  $I_\varepsilon = \langle z^{W_\varepsilon} - 1 \rangle : \langle z - 1 \rangle$ ,  $\varepsilon \in \{0, 1\}$ , coincide and have the  $H$ -basis representation*

$$(4.5) \quad I := I_0 = I_1 = \langle (z_1^3 + z_1^2 + z_1 + 1)(z_2 + 1) \rangle + \langle z_1^4 - 1 \rangle + \langle z_2^2 - 1 \rangle.$$

The fact that  $I_0 = I_1$  may appear a little bit surprising at first view, since it implies that, for any finitely supported mask  $a$ , we have

$$\sum_{\beta \in \mathbb{Z}^2} a(\alpha + W_0 \beta) = 1, \quad \alpha \in \mathbb{Z}^2 \quad \Leftrightarrow \quad \sum_{\beta \in \mathbb{Z}^2} a(\alpha + W_1 \beta) = 1, \quad \alpha \in \mathbb{Z}^2.$$

Hence the necessary ‘‘sum rule’’ condition with respect to  $W_0$  is equivalent to the one with respect to  $W_1$ . However, if we write  $W_1 = W_0 V$  with the unimodular matrix  $V = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ , then a simple change of the summation variable indeed gives for any  $\alpha \in \mathbb{Z}^2$

$$\sum_{\beta \in \mathbb{Z}^2} a(\alpha + W_1 \beta) = \sum_{\beta \in \mathbb{Z}^2} a(\alpha + W_0 V \beta) = \sum_{\beta \in \mathbb{Z}^2} a(\alpha + W_0 \beta),$$

and confirms (4.5).

Moreover, note that Theorem 4.12 gives a way to *parameterize* the ideal of all admissible polynomial masks. Indeed, for any  $n \in \mathbb{N}$  we have that

$$\begin{aligned} I \cap \Pi_d &= p(z)(z_1^4 - 1) + q(z)(z_1^3 + z_1^2 + z + 1)(z_2 + 1) + r(z)(z_2^2 - 1), \\ &\text{with } \deg p \leq n - 4, \deg q \leq n - 4, \text{ and } \deg r \leq n - 2. \end{aligned}$$

For a polynomial of this form, the decomposition with respect to  $W_0$ , i.e., the matrix polynomial  $B_0$ , becomes

$$(4.6) \quad B_0^*(z) = \begin{pmatrix} (z_1 - 1)p(z) + (z_2 + 1)q(z) & (z_1 - 1)r(z) \\ (z_2 - 1)p(z) & (z_1^3 + z_1^2 + z_1 + 1)q(z) + (z_2 - 1)r(z) \end{pmatrix}$$

Since the two Laurent ideals  $I_0$  and  $I_1$  coincide, the decomposition of  $a_1^*$  into  $B_1^*$  takes exactly the same form as  $B_0^*$  in (4.6).

Next, we rephrase the identity (4.4) by means of the *backwards difference operator*  $\nabla$ , defined for a sequence  $a$  as

$$\nabla a := \begin{pmatrix} a(\cdot - \eta_1) - a(\cdot) \\ a(\cdot - \eta_2) - a(\cdot) \end{pmatrix}, \quad (\nabla a)^*(z) = [z - 1] a^*(z),$$

where  $\eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $\eta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  denote the unit multiindices in  $\mathbb{Z}^2$ . Since, in addition, any finitely supported matrix sequence  $B$  satisfies

$$(S_{B, W_\varepsilon} c)^*(z) = B^*(z) c(z^{W_\varepsilon}), \quad c \in \nabla \ell_\infty(\mathbb{Z}^2), \quad \varepsilon \in \{0, 1\},$$

where

$$S_{B, W_\varepsilon} c := \sum_{\alpha \in \mathbb{Z}^2} B(\cdot - W_\varepsilon \alpha) c(\alpha),$$

our quotient ideal representation (4.4) can equivalently be written in terms of the difference operator as

$$\nabla S_{a_\varepsilon, W_\varepsilon} = S_{B_\varepsilon, W_\varepsilon} \nabla, \quad \varepsilon \in \{0, 1\}.$$

We end this section by recalling that quotient ideal containment also characterizes the order of *polynomial reproduction* provided by the two masks and thus the subdivision scheme. Recall that a mask  $a$  provides polynomial reproduction of order  $n$ , if the leading forms of all polynomial sequences are reproduced by the scheme:

$$S_a \Pi_k = \Pi_k, \quad k = 0, \dots, n, \quad \Pi_k := \left\{ \sum_{|\gamma| \leq k} a_\gamma \alpha^\gamma : \alpha \in \mathbb{Z}^2 \right\}.$$

Polynomial reproduction is essential for the smoothness of the refinable limit function [6] as well as for the approximation order of the associated wavelet construction. With the methods from [26, 28] we can now easily describe polynomial reproduction.

**Theorem 4.13.** *The directional subdivision scheme preserves polynomials of degree  $n$ , i.e.,  $S_\varepsilon \Pi_k = \Pi_k$ ,  $\varepsilon \in E$ ,  $k = 0, \dots, n$ , if and only if*

$$a_\varepsilon \in I^{n+1} = (\langle z^{W_0} - 1 \rangle : \langle z - 1 \rangle)^{n+1} = \langle z^{W_0} - 1 \rangle^{n+1} : \langle z - 1 \rangle^{n+1}.$$

**4.3. A Characterization of Convergence.** Finally, we will give a characterization of convergence of the adaptive directional subdivision scheme, like usually in terms of a (restricted joint) spectral radius. In this subsection, the adaptive directional subdivision scheme both for masks  $a_0$  and  $a_1$  as well as for their associated matrix sequences  $B_0$  and  $B_1$  will come into play. To distinguish both, for the first, we again employ the notation  $S_\varepsilon$ ,  $\varepsilon \in E$ , whereas the second adaptive directional subdivision scheme will be denoted by  $S_\varepsilon^B$ ,  $\varepsilon \in E$ .

Now, given two matrix masks  $B_\varepsilon$ ,  $\varepsilon \in \{0, 1\}$ , their *restricted joint spectral radius* is defined as

$$\rho(B_0, B_1 | \nabla) = \limsup_{n \rightarrow \infty} \sup_{\varepsilon \in \{0, 1\}^n} \sup_{c \in \nabla \ell_\infty} \|S_\varepsilon^B c\|_\infty^{1/n}.$$

The joint spectral radius is called “restricted” since the supremum is not taken over all 2–vector valued sequences but only over the proper subset  $\nabla \ell_\infty$ , see [8, 30]. The main result of this paragraph is now as follows.

**Theorem 4.14.** *The adaptive directional subdivision scheme based on the masks  $a_\varepsilon$ ,  $\varepsilon \in \{0, 1\}$  converges if and only if  $a_\varepsilon^*(z) \in I$  and the representation masks  $B_\varepsilon$ ,  $\varepsilon \in \{0, 1\}$ , satisfy  $\rho(B_0, B_1 | \nabla) < 1$ .*

We will split the lengthy proof of Theorem 4.14 into several partial results, beginning with the sufficiency of the spectral radius condition. To that end, we will show that, starting with a particular test function  $g$ , the sequence  $g * S_{P_n, \varepsilon} c$  converges to a limit function for any choice of  $\varepsilon \in E_\infty$  and any  $c \in \ell_\infty$ . Indeed, we choose the test function  $g$  to be  $W_0$ -refinable with respect to a mask  $b$ , that is

$$(4.7) \quad g = \sum_{\alpha \in \mathbb{Z}^2} b(\alpha) g(W_0 \cdot -\alpha).$$

Such functions can be easily shown to exist, even with an arbitrary order of smoothness: pick any cardinal B-spline  $\phi = M(\cdot | 0, \dots, N)$  with refinement mask  $h$ , then a double application of the refinement equation with respect to the first variable shows that the tensor product function

$$g(x, y) = [(\phi * \phi) \otimes \phi](x, y) = (\phi * \phi)(x) \phi(y) =: \psi(x) \phi(y)$$

is  $W_0$ -refinable with respect to the mask  $b = S_{h, 2} h \otimes h$ , where  $S_{h, 2}$  denotes the subdivision scheme with mask  $h$  and dilation 2. The following lemma states a more general process.

**Lemma 4.15.** *Let  $b_1, b_2 \in \ell(\mathbb{Z})$  be 2-refinable masks, and let the mask  $\tilde{b}_1$  be defined by  $\tilde{b}_1(m) = S_{b_1, 2} b_1(m) = \sum_{k \in \mathbb{Z}} b_1(k) b_1(m - 2k)$ . Then the mask  $a_0 = \tilde{b}_1 \otimes b_2$  is  $W_0$ -refinable, and  $a_1 = a_0(U \cdot)$  is  $W_1$ -refinable.*

*Proof.* Let  $\varphi_1, \varphi_2$  be univariate functions which are 2-refinable with respect to  $b_1, b_2$ , respectively, i.e.,

$$\varphi_i = \sum_{k \in \mathbb{Z}} b_i(k) \varphi(2 \cdot -k), \quad i = 1, 2.$$

We claim that the function  $f$  defined by

$$f = \varphi_1 \otimes \varphi_2$$

is  $W_0$ -refinable with respect to  $a_0$ . Indeed, for  $x = (x_1, x_2) \in \mathbb{R}^2$ , we obtain

$$\begin{aligned} & \sum_{\alpha \in \mathbb{Z}^2} (\tilde{b}_1 \otimes b_2)(\alpha) f(W_0 x - \alpha) \\ &= \left[ \sum_{\alpha_1 \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} b_1(\alpha_1 - 2k) b_1(k) \right) \varphi_1(4x_1 - \alpha_1) \right] \left[ \sum_{\alpha_2 \in \mathbb{Z}} b_2(\alpha_2) \varphi_2(2x_2 - \alpha_2) \right] \\ &= \left[ \sum_{k \in \mathbb{Z}} b_1(k) \sum_{\alpha_1 \in \mathbb{Z}} b_1(\alpha_1) \varphi_1(4x_1 - 2k - \alpha_1) \right] \varphi_2(x_2) \\ &= \left[ \sum_{k \in \mathbb{Z}} b_1(k) \varphi_1(2x_1 - k) \right] \varphi_2(x_2) \\ &= f(x). \end{aligned}$$

The claim concerning  $W_1$ -refinability of  $a_1$  follows from Lemma 4.16.  $\square$

There also exists a canonical  $W_1$ -refinable function associated to  $g$ .

**Lemma 4.16.** *If  $g_0 = g$  is  $W_0$ -refinable with respect to the mask  $b_0 = b$ , then  $g_1 = g_0(U \cdot)$  is  $W_1$ -refinable with respect to the mask  $b_1 = b_0(U \cdot)$ .*

*Proof.* Setting  $g_1 = g_0(U \cdot)$  and thus  $g_0 = g_1(U^{-1} \cdot)$ , we find for  $x \in \mathbb{R}^2$  that

$$\begin{aligned} g_1(x) &= g_0(Ux) = \sum_{\alpha \in \mathbb{Z}^2} b_0(\alpha) g_0(W_0 Ux - \alpha) = \sum_{\alpha \in \mathbb{Z}^2} b_0(\alpha) g_0(W_0 V^2 x - \alpha) \\ &= \sum_{\alpha \in \mathbb{Z}^2} b_0(\alpha) g_1(U^{-1} W_1 Vx - U^{-1} \alpha) = \sum_{\alpha \in \mathbb{Z}^2} b_0(U\alpha) g_1(W_1 x - \alpha) \\ &= \sum_{\alpha \in \mathbb{Z}^2} b_1(\alpha) g_1(W_1 x - \alpha), \end{aligned}$$

hence  $g_1$  is  $W_1$ -refinable with respect to  $b_1$ .  $\square$

The next two observations are again of a more algebraic nature.

**Lemma 4.17.** *Suppose that a mask  $a$  satisfies  $S_{a, W_\varepsilon} c = 0$  for all constant sequences  $c$  and some  $\varepsilon \in \{0, 1\}$ . Then there exists a  $1 \times 2$  matrix mask  $B$  such that  $S_{a, W_\varepsilon} = S_{B, W_\varepsilon} \nabla$ .*

*Proof.* Again we refer to [26, 28] where it has been shown that  $S_{a, W_\varepsilon} c = 0$  for all constant sequences  $c$  if and only if  $a^*(z) \in \langle z^{W_\varepsilon} - 1 \rangle$  which is in turn equivalent to the existence of a representation

$$a^*(z) = b_1^*(z) \left( z^{(W_\varepsilon)_1} - 1 \right) + b_2^*(z) \left( z^{(W_\varepsilon)_2} - 1 \right) = B^*(z) [z^{W_\varepsilon} - 1],$$

which is nothing but  $S_{a, W_\varepsilon} = S_{B, W_\varepsilon} \nabla$ .  $\square$

**Lemma 4.18.** *Suppose that a compactly supported function  $f$  satisfies  $f * c = 0$  for all constant sequences, then there exists a compactly supported, continuous  $1 \times 2$  matrix function  $G$  such that  $f * c = G * \nabla c$  for all  $c \in \ell_\infty(\mathbb{Z}^2)$ .*

*Proof.* For any  $x \in [0, 1]^2$  we consider the sequence  $f_x = (f(x + \alpha) : \alpha \in \mathbb{Z}^2)$ . Since  $f$  is compactly supported, any such sequence  $f_x$ ,  $x \in [0, 1]^2$  has finite support and since  $f$  is continuous, the map  $x \mapsto f_x$  is a continuous one.

By assumption,  $f_x * c = 0$  for any  $x$  and any constant sequence  $c$ , hence, with the scaling matrix  $I$ , the same methods as above yield that  $f_x^* \in \langle z^I - 1 \rangle = \langle z - 1 \rangle$ . Consequently, we have that

$$f_x^*(z) = g_{x,1}^*(z) (z_1 - 1) + g_{x,2}^*(z) (z_2 - 1) = G_x^*(z) [z - 1]$$

where, like  $f_x$  and  $f_x^*(z)$ , also  $G_x^*(z)$  depend continuously on  $x$  as they can be obtained by applying the orthogonal reduction process from [27]. Therefore, the function  $G$ , defined as

$$G(x + \alpha) = G_x(\alpha), \quad x \in [0, 1]^2, \quad \alpha \in \mathbb{Z}^2$$

has the properties claimed in the statement of the lemma.  $\square$

Now we are in position to prove the sufficiency of the spectral radius condition which we state as a separate proposition.

**Proposition 4.19.** *The adaptive directional subdivision scheme based on the masks  $a_\varepsilon$ ,  $\varepsilon \in \{0, 1\}$  converges, if  $a_\varepsilon^*(z) \in I$  and the representation masks  $B_\varepsilon$ ,  $\varepsilon \in \{0, 1\}$ , satisfy  $\rho(B_0, B_1 | \nabla) < 1$ .*

*Proof.* For any  $\theta \in (0, 1 - \rho)$ , there exists, by standard properties of the (joint) spectral radius, a constant  $C > 0$  such that

$$\|S_{P_n \varepsilon}^B \nabla c\|_\infty \leq C(\rho + \theta)^n = C\sigma^n, \quad n \in \mathbb{N}, \quad \varepsilon \in E_\infty, \quad c \in \ell_\infty(\mathbb{Z}^2),$$

where  $0 < \sigma := \rho + \theta < 1$ .

Now, let  $\varepsilon \in E_\infty$  be given and suppose first that  $\varepsilon_n = 0$ . Then, by the refinability of the test function  $g$  from (4.7) and Lemma 4.17 which ensures the existence of a finitely supported matrix mask  $F$  such that  $S_0 - S_{b, W_0} = S_{F, W_0} \nabla$ , we have that

$$\begin{aligned} & \|g * S_{P_n \varepsilon} c(W_{P_n \varepsilon} \cdot) - g * S_{P_{n-1} \varepsilon} c(W_{P_{n-1} \varepsilon} \cdot)\|_\infty \\ &= \|g * S_{P_n \varepsilon} c(W_{P_n \varepsilon} \cdot) - g * S_{b, W_0} S_{P_{n-1} \varepsilon} c(W_{P_n \varepsilon} \cdot)\|_\infty \\ &= \|g * (S_0 - S_{b, W_0}) S_{P_{n-1} \varepsilon} c\|_\infty \\ &\leq B_g \|(S_0 - S_{b, W_0}) S_{P_{n-1} \varepsilon} c\|_\infty = B_g \|S_{F, W_0} \nabla S_{P_{n-1} \varepsilon} c\|_\infty \\ &= B_g \|S_{F, W_0} S_{P_{n-1} \varepsilon}^B \nabla c\|_\infty \leq B_g \|S_{F, W_0}\| C \sigma^{n-1}. \end{aligned}$$

If on the other hand  $\varepsilon_n = 1$ , by using the function  $g_1 = g(U \cdot)$  (cf. Lemma 4.16) we pass to the estimate

$$\begin{aligned} & \|g * S_{P_n \varepsilon} c(W_{P_n \varepsilon} \cdot) - g * S_{P_{n-1} \varepsilon} c(W_{P_{n-1} \varepsilon} \cdot)\|_\infty \\ &\leq \|(g - g_1) * S_{P_n \varepsilon} c(W_{P_n \varepsilon} \cdot)\|_\infty + \|(g - g_1) * S_{P_{n-1} \varepsilon} c(W_{P_{n-1} \varepsilon} \cdot)\|_\infty \\ &\quad + \|g_1 * S_{P_n \varepsilon} c(W_{P_n \varepsilon} \cdot) - g_1 * S_{P_{n-1} \varepsilon} c(W_{P_{n-1} \varepsilon} \cdot)\|_\infty. \end{aligned}$$

For the first two terms we now make use of Lemma 4.18 to obtain that

$$\begin{aligned} & \|(g - g_1) * S_{P_n \varepsilon} c(W_{P_n \varepsilon} \cdot)\|_\infty \\ &= \|(g - g_1) * S_{P_n \varepsilon} c\|_\infty = \|G * \nabla S_{P_n \varepsilon} c\|_\infty = \|G * S_{P_n \varepsilon}^B \nabla c\|_\infty \leq B_G C \sigma^n \end{aligned}$$

and

$$\|(g - g_1) * S_{P_{n-1} \varepsilon} c(W_{P_{n-1} \varepsilon} \cdot)\|_\infty \leq B_G C \sigma^{n-1},$$

respectively, while the third term can now be estimated as above again. In summary, we obtain that there exists a constant  $D > 0$  such that

$$\|g * S_{P_n \varepsilon} c(W_{P_n \varepsilon} \cdot) - g * S_{P_{n-1} \varepsilon} c(W_{P_{n-1} \varepsilon} \cdot)\|_\infty \leq D \sigma^{n-1}$$

so that for  $m \in \mathbb{N}$

$$\|g * S_{P_{n+m} \varepsilon} c(W_{P_{n+m} \varepsilon} \cdot) - g * S_{P_n \varepsilon} c(W_{P_n \varepsilon} \cdot)\|_\infty \leq D \frac{\sigma^n}{1 - \sigma}.$$

In other words, the sequence  $g * S_{P_n \varepsilon} c(W_{P_n \varepsilon} \cdot)$  is a Cauchy sequence of continuous functions and thus must converge to a limit function for  $n \rightarrow \infty$ . Convergence of the subdivision scheme then follows by standard means.  $\square$

The proof of the converse statement of Proposition 4.19 is based on the estimate

$$\begin{aligned} \|S_{P_n \varepsilon}^B \nabla \delta\|_\infty &= \|\nabla S_{P_n \varepsilon} \delta\|_\infty = \left\| \begin{pmatrix} S_{P_n \varepsilon} \delta(\cdot - \eta_1) - S_{P_n \varepsilon} \delta(\cdot) \\ S_{P_n \varepsilon} \delta(\cdot - \eta_2) - S_{P_n \varepsilon} \delta(\cdot) \end{pmatrix} \right\|_\infty \\ &= \max_{j=1,2} \|S_{P_n \varepsilon} \delta(\cdot - \eta_j) - S_{P_n \varepsilon} \delta(\cdot)\|_\infty \\ &\leq \max_{j=1,2} \left( \|S_{P_n \varepsilon} \delta(\cdot - \eta_j) - \sigma_{W_{P_n \varepsilon}^{-1}} f(\cdot - \eta_j)\|_\infty + \|S_{P_n \varepsilon} \delta - \sigma_{W_{P_n \varepsilon}^{-1}} f\|_\infty \right. \\ &\quad \left. + \|\sigma_{W_{P_n \varepsilon}^{-1}} f(\cdot - \eta_j) - \sigma_{W_{P_n \varepsilon}^{-1}} f(\cdot)\|_\infty \right), \end{aligned}$$

hence,

$$\|S_{P_n^\varepsilon}^B \nabla \delta\|_\infty \leq \max_{j=1,2} \left( 2 \|S_{P_n^\varepsilon} \delta - \sigma_{W_{P_n^\varepsilon}^{-1}} f\|_\infty + \|\sigma_{W_{P_n^\varepsilon}^{-1}} f(\cdot - \eta_j) - \sigma_{W_{P_n^\varepsilon}^{-1}} f(\cdot)\|_\infty \right).$$

If we assume that the subdivision scheme converges with uniformly continuous limit function, then the right hand side converges to zero, hence also  $\|S_{P_n^\varepsilon}^B \nabla c\|_\infty \rightarrow 0$  for  $n \rightarrow \infty$  and any  $c \in \ell_\infty(\mathbb{Z}^2)$ . This, however, is not sufficient for our purposes. To show that the restricted spectral radius of  $\rho(B_0, B_1 | \nabla)$  is less than one, we have to show that

$$(4.8) \quad \|S_{P_n^\varepsilon}^B \nabla c\|_\infty \leq C \theta_n \|\nabla c\|_\infty, \quad \lim_{n \rightarrow \infty} \theta_n = 0,$$

which will be prepared in the next lemmas. Here we follow the outline of a proof from [6] and show that there exists a constant  $C > 0$  such that

$$\|\nabla S_{P_n^\varepsilon} c\|_\infty \leq C \theta_n \|\nabla c\|_\infty, \quad \lim_{n \rightarrow \infty} \theta_n = 0,$$

from which (4.8) follows immediately. We begin with an estimate on the limit function  $f_\varepsilon$ .

**Lemma 4.20.** *If  $a_0$  and  $a_1$  define a convergent subdivision scheme, then there exists a constant  $C_1 > 0$  such that for any  $\varepsilon \in E_\infty$  and any  $c \in \ell_\infty(\mathbb{Z}^2)$*

$$|f_\varepsilon * c(x) - f_\varepsilon * c(y)| \leq C_1 \omega(f_\varepsilon, \delta) \|\nabla c\|_\infty, \quad \|x - y\| \leq \delta < 1.$$

*Proof.* Since, according to Lemma 4.3, convergence implies the preservation of constant sequences by the subdivision scheme, we also have that

$$1 = f_\varepsilon * 1 = \sum_{\alpha \in \mathbb{Z}^2} f_\varepsilon(\cdot - \alpha)$$

and thus, for any  $c \in \ell_\infty$ , any  $w \in \mathbb{R}$  and any  $x, y \in \mathbb{R}^2$  with  $\|x - y\| \leq \delta$ ,

$$\begin{aligned} |f_\varepsilon * c(x) - f_\varepsilon * c(y)| &= \left| \sum_{\alpha \in \mathbb{Z}^2} (f_\varepsilon(x - \alpha) - f_\varepsilon(y - \alpha)) (c(\alpha) - w) \right| \\ &\leq \#\Omega_{x,y} \cdot \omega(f_\varepsilon, \delta) \cdot \max_{\alpha \in \Omega_{x,y}} |c(\alpha) - w|, \end{aligned}$$

where

$$\Omega_{x,y} = \{\alpha \in \mathbb{Z}^2 : f_\varepsilon(x - \alpha) \neq 0\} \cup \{\alpha \in \mathbb{Z}^2 : f_\varepsilon(y - \alpha) \neq 0\}.$$

Since  $f_\varepsilon$  is finitely supported, we have that  $\#\Omega_{x,y} < \infty$ . Specifically, if we assume that  $f_\varepsilon$  is supported on  $[-N, N]^2$ , then  $\#\Omega_{x,y} \leq (2N + 2)^2$  as long as  $\delta < 1$ . Choosing

$$w = \frac{1}{2} \left( \max_{\alpha \in \Omega_{x,y}} c(\alpha) + \min_{\alpha \in \Omega_{x,y}} c(\alpha) \right),$$

it follows for any  $\alpha \in \Omega_{x,y}$  that

$$|c(\alpha) - w| \leq \frac{1}{2} \left| \max_{\alpha \in \Omega_{x,y}} c(\alpha) + \min_{\alpha \in \Omega_{x,y}} c(\alpha) \right| \leq \frac{1}{2} \#\Omega_{x,y} \|\nabla c\|_\infty,$$

hence,

$$|f_\varepsilon * c(x) - f_\varepsilon * c(y)| \leq \frac{1}{2} (2N + 2)^4 \omega(f_\varepsilon, \delta) \|\nabla c\|_\infty$$

as claimed.  $\square$

The next result concerns the difference between the subdivision scheme and the limit function.

**Lemma 4.21.** *If the adaptive directional subdivision scheme based on the masks  $a_\varepsilon$ ,  $\varepsilon \in \{0, 1\}$ , then there exists a constant  $C_2 > 0$  such that, for any  $n \in \mathbb{N}$ , we have*

$$\|S_{P_n \varepsilon} c - f_\varepsilon * c(W_{P_n \varepsilon}^{-1} \cdot)\|_\infty \leq C_2 \|S_{P_n \varepsilon} \delta - f_\varepsilon(W_{P_n \varepsilon}^{-1} \cdot)\|_\infty \|\nabla c\|_\infty.$$

*Proof.* We fix  $n$ , set, for abbreviation,  $\hat{\varepsilon} = P_n \varepsilon$ , and assume again that  $f_\varepsilon$  as well as  $a_0$  and  $a_1$  are supported on  $[-N, N]^2$ . Again, we make use of the fact that  $S_{\hat{\varepsilon}}$  and  $f_\varepsilon$  preserve constant data and obtain, for any  $\alpha \in \mathbb{Z}^2$  and  $w \in \mathbb{R}$ , that

$$S_{\hat{\varepsilon}} c(\alpha) - f_\varepsilon * c(W_{\hat{\varepsilon}}^{-1} \alpha) = \sum_{\beta \in \mathbb{Z}^2} (a_{\hat{\varepsilon}}(\alpha - W_{\hat{\varepsilon}} \beta) - f_\varepsilon(W_{\hat{\varepsilon}}^{-1} \alpha - \beta)) (c(\beta) - w).$$

Since

$$\Omega_{\alpha, \hat{\varepsilon}} = \{\alpha \in \mathbb{Z}^2 : a_{\hat{\varepsilon}}(\alpha - W_{\hat{\varepsilon}} \beta) \neq 0\} \cup \{\alpha \in \mathbb{Z}^2 : f_\varepsilon(W_{\hat{\varepsilon}}^{-1} \alpha - \beta) \neq 0\}$$

again satisfies  $\#\Omega_{\alpha, \hat{\varepsilon}} \leq (2N + 2)^2$ , the same judicious choice of  $w$  as above leads to the estimate

$$|S_{\hat{\varepsilon}} c(\alpha) - f_\varepsilon * c(W_{\hat{\varepsilon}}^{-1} \alpha)| \leq (2N + 2)^4 \sup_{\alpha \in \mathbb{Z}^2} |a_{\hat{\varepsilon}}(\alpha) - f_\varepsilon(W_{\hat{\varepsilon}}^{-1} \alpha)| \|\nabla c\|_\infty,$$

from which the claim follows immediately.  $\square$

Now it is easy to complete the proof of the converse statement for convergence which we formulate in the following way.

**Proposition 4.22.** *If the adaptive directional subdivision scheme based on the masks  $a_\varepsilon$ ,  $\varepsilon \in \{0, 1\}$  converges then  $a_\varepsilon^*(z) \in I$  and the representation masks  $B_\varepsilon$ ,  $\varepsilon \in \{0, 1\}$ , satisfy  $\rho(B_0, B_1 | \nabla) < 1$ .*

*Proof.* In Lemma 4.3, it has already been shown that convergence implies  $a_\varepsilon^*(z) \in I$ . Moreover, Lemma 4.20 and Lemma 4.21 allow us to conclude with  $C = \max\{C_1, C_2\}$  that, for any  $c \in \ell_\infty$ , we have

$$\begin{aligned} \|S_{P_n \varepsilon}^B \nabla c\|_\infty &= \|\nabla S_{P_n \varepsilon} c\|_\infty \\ &\leq \|\nabla (S_{P_n \varepsilon} c - f_\varepsilon * c(W_{P_n \varepsilon}^{-1} \cdot))\|_\infty + \|\nabla f_\varepsilon * c(W_{P_n \varepsilon}^{-1} \cdot)\|_\infty \\ &\leq 2C \left( \|S_{P_n \varepsilon} \delta - f_\varepsilon(W_{P_n \varepsilon}^{-1} \cdot)\|_\infty + \omega(f_\varepsilon, \|W_{P_n \varepsilon}^{-1}\|) \right) \|\nabla c\|_\infty, \end{aligned}$$

and since

$$\lim_{n \rightarrow \infty} \|S_{P_n \varepsilon} \delta - f_\varepsilon(W_{P_n \varepsilon}^{-1} \cdot)\|_\infty + \omega(f_\varepsilon, \|W_{P_n \varepsilon}^{-1}\|) = 0$$

by convergence of the adaptive directional subdivision scheme and uniform continuity of the limit function, our prove is complete.  $\square$

## 5. NUMERICAL EXPERIMENTS

In this section we present some numerical experiments which illustrate the ability of the developed class of subdivision schemes to adaptively change the orientation of the data.

First, we recall that there exist a general way to construct masks, which are refinable with respect to the dilation matrices  $W_0$  and  $W_1$ , compare Lemma 4.15. Now let the mask  $b \in \ell(\mathbb{Z})$  be chosen by  $b(-3) = -\frac{1}{16} = b(3)$ ,  $b(-1) = \frac{9}{16} =$

$b(1)$ ,  $b(0) = 1$  and  $b(m) = 0$  otherwise, which coincides with the mask studied by Deslauriers and Dubuc [15]. We remark that this mask yields a 2-interpolatory subdivision scheme (compare also Section 6). By Lemma 4.15, we know that  $a_0 = \tilde{b} \otimes b$  is  $W_0$ -refinable, and  $a_1 = a_0(U \cdot)$  is  $W_1$ -refinable.

In Figure 4 we illustrate the refinement of the matrix

$$(5.1) \quad C_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

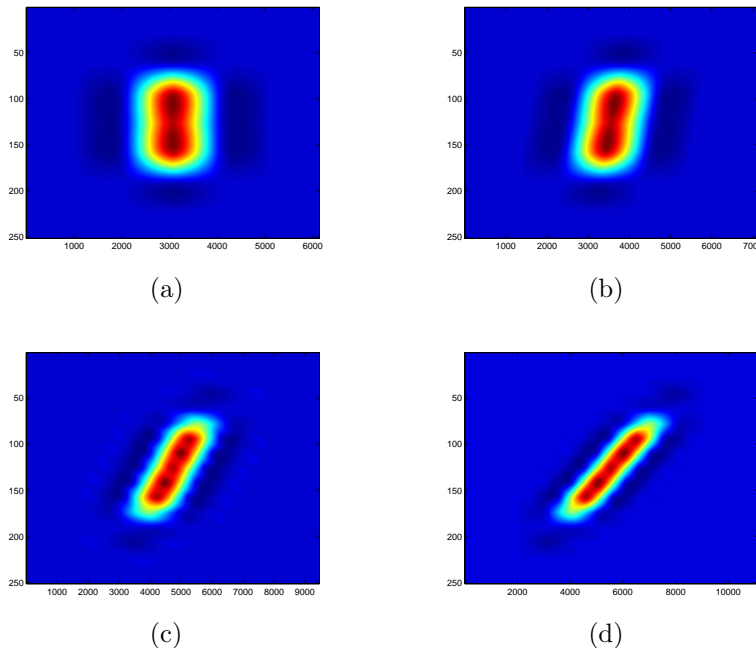


FIGURE 4. This figure shows the refinement of the matrix  $C_1$  defined in (5.1) after applying  $S_\varepsilon$  with (a)  $\varepsilon = (0, 0, 0, 0, 0)$ , (b)  $\varepsilon = (0, 0, 0, 1, 0)$ , (c)  $\varepsilon = (0, 1, 0, 0, 0)$ , and (d)  $\varepsilon = (0, 1, 1, 1, 1)$ .

and in Figure 5 we subdivide the data given by

$$(5.2) \quad C_2 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

In both figures we employ different iterations of the subdivision schemes  $S_0$  and  $S_1$ . As can clearly be seen, the application of  $S_1$  increases the angle the resulting images is sheared in the  $x$ -direction, where the angle depends on the particular path in the binary tree (see Figure 2) we choose.

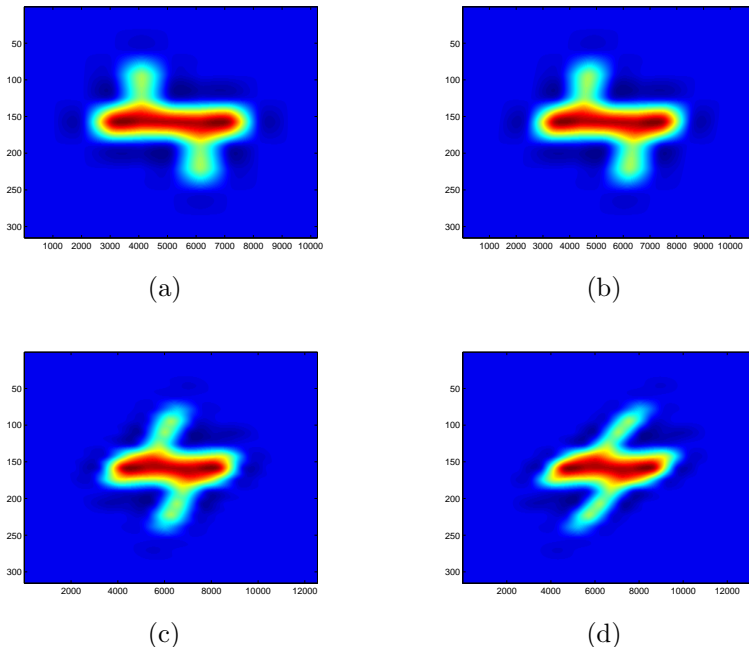


FIGURE 5. This figure shows the refinement of the matrix  $C_2$  defined in (5.2) after applying  $S_\varepsilon$  with (a)  $\varepsilon = (0, 0, 0, 0, 0)$ , (b)  $\varepsilon = (0, 0, 0, 1, 0)$ , (c)  $\varepsilon = (0, 1, 0, 0, 0)$ , and (d)  $\varepsilon = (0, 1, 1, 1, 1)$ .

## 6. SHEARLET MULTIREOLUTION ANALYSIS

In this section we will show how the adaptive directional subdivision schemes developed in the previous sections can be applied to derive a shearlet multiresolution analysis. For the sake of simplicity, in the computation of “dual functions” we will restrict ourselves to interpolatory subdivision schemes in this paper. Our idea is inspired by similar ideas for the construction of a fast wavelet decomposition from interpolatory subdivision schemes [17]. The construction of a shearlet multiresolution analysis associated with general adaptive directional subdivision schemes is beyond the scope of this paper, and will be studied in a forthcoming paper.

Before constructing the scaling spaces we first need to discuss whether there exist masks  $a_0$  and  $a_1$  such that the subdivision schemes  $S_0$  and  $S_1$  are both interpolatory, respectively, which immediately implies that  $S_\varepsilon$  is interpolatory for each  $\varepsilon \in E_\infty$ . To that end, we proceed by using a tensor product approach. Recall that a mask  $a_0$  leads to an interpolatory subdivision scheme  $S_0$  provided that

$$(6.1) \quad a_0(W_0\alpha) = \delta_{\alpha,0} \quad \text{for all } \alpha \in \mathbb{Z}^2,$$

likewise does a mask  $a_1$  lead to an interpolatory subdivision scheme  $S_1$  provided that

$$(6.2) \quad a_1(W_1\alpha) = \delta_{\alpha,0} \quad \text{for all } \alpha \in \mathbb{Z}^2.$$

There exists a canonical way to define  $a_1$  by means of the matrix  $U$  as indicated by the following lemma (compare also Lemma 4.15).

**Lemma 6.1.** *Let  $b_1, b_2 \in \ell(\mathbb{Z})$  be masks which satisfy  $b_i(2m) = \delta_{m,0}$  for all  $m \in \mathbb{Z}$ ,  $i = 1, 2$  and let the mask  $\tilde{b}_1$  be defined by  $\tilde{b}_1(m) = S_{b_1, 2} b_1(m) = \sum_{k \in \mathbb{Z}} b_1(k) b_1(m - 2k)$ . Then the mask  $\tilde{b}_1 \otimes b_2$  satisfies (6.1), and the mask  $(\tilde{b}_1 \otimes b_2)(U \cdot)$  satisfies (6.2).*

*Proof.* Given some  $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$ , we obtain

$$(\tilde{b}_1 \otimes b_2)(W_0 \alpha) = \sum_{k \in \mathbb{Z}} b_1(k) b_1(4\alpha_1 - 2k) \cdot \delta_{2\alpha_2, 0} = \sum_{k \in \mathbb{Z}} b_1(k) \delta_{2\alpha_1 - k, 0} \cdot \delta_{\alpha_2, 0} = \delta_{\alpha, 0}.$$

A similar computation shows  $(\tilde{b} \otimes b)(UW_1 \alpha) = \delta_{\alpha, 0}$ .  $\square$

Suppose we have chosen masks  $a_0$  and  $a_1$  so that the subdivision scheme  $S_\varepsilon$  is interpolatory and converges for each  $\varepsilon \in E_\infty$ . To define the scaling functions, recall that we wrote  $\varepsilon^* = (\varepsilon, 0, 0 \dots)$  for the canonical embedding of  $E$  into  $E_\infty$ ; the image of this embedding operation,

$$E^* = \{\varepsilon^* : \varepsilon \in E\} \subset E_\infty$$

thus consists of all infinite 0-1-sequences which contain only a finite number of nonzero components. It is worthwhile to keep in mind that the subdivision scheme  $S_\varepsilon$  converges for all  $\varepsilon \in E^*$  if and only if  $a_0$  defines a convergent subdivision scheme and hence the functions

$$\{f_\varepsilon : \varepsilon \in E^*\} = \{f_{\varepsilon^*} : \varepsilon \in E\}$$

which will be needed to build the MRA can be ensured to exist by requiring the existence of an appropriate solution of the refinement equation associated to  $a_0$ . This is a much weaker condition, of course, than convergence of the  $S_\varepsilon$  for any  $\varepsilon \in E_\infty$ .

**Definition 6.2.** The *shearlet scaling spaces* are defined as

$$V_0 = \text{span} \{f_{\varepsilon^*}(\cdot - \alpha) : \alpha \in \mathbb{Z}^2, \varepsilon \in E\}$$

and

$$V_n = \sum_{\varepsilon \in \{0,1\}^n} V_\varepsilon, \quad n \geq 1,$$

where

$$V_\varepsilon = \text{span} \{f(W_\varepsilon \cdot - \alpha) : \alpha \in \mathbb{Z}^2, f \in V_0\} \quad \text{for all } \varepsilon \in E.$$

Indeed this choice of scaling spaces provides a multiresolution analysis, which is the focus of the following theorem. The main ingredient in the proof is – as it should be – the refinement equation (4.3).

**Theorem 6.3.** *The spaces  $(V_n)_{n \geq 0}$  create a multiresolution analysis. In particular,*

- (i) *the spaces  $V_n$ ,  $n \geq 0$  are translation invariant,*
- (ii)  *$V_n \subseteq V_{n+1}$  for all  $n \geq 0$ , and*
- (iii) *for each  $n \in \mathbb{N}$ , we have  $f \in V_n \Leftrightarrow f(W_\varepsilon \cdot) \in V_{n+1}$  for each  $\varepsilon \in \{0, 1\}$ .*

*Proof.* Statement (i) follows immediately from the definition of  $V_n$ , which is a translational completion.

To verify the nestedness property (ii), we consider an arbitrary “basis element”  $f \in V_n$  of the form

$$(6.3) \quad f = f_{\eta^*}(W_\varepsilon \cdot - \alpha), \quad \varepsilon \in E_n, \quad \eta = (\eta_1, \hat{\eta}) \in E, \quad \alpha \in \mathbb{Z}^2,$$

and make use of the refinement equation (4.3) to verify that

$$f = \sum_{\beta \in \mathbb{Z}^2} a_{\eta_1}(\beta) f_{\hat{\eta}^*}(W_{\eta_1}(W_\varepsilon \cdot -\alpha) - \beta) = \sum_{\beta \in \mathbb{Z}^2} a_{\eta_1}(\beta - W_{\eta_1}\alpha) f_{\hat{\eta}^*}(W_{\varepsilon'} \cdot -\beta),$$

with  $\varepsilon' = (\varepsilon, \eta_1) \in E_{n+1}$ , hence  $f \in V_{\varepsilon'} \subseteq V_{n+1}$ .

To verify (iii) we again consider a function element  $f \in V_n$  of the form (6.3). One implication follows from

$$f(W_\tau \cdot) = f_{\eta^*}(W_{(\tau, \varepsilon)} \cdot -\alpha), \quad \tau \in \{0, 1\},$$

the other one can be deduced in a similar way by considering  $f \in V_{n+1}$  and showing that this yields  $f(W_\tau^{-1} \cdot) \in V_n$  for any  $\tau \in \{0, 1\}$ .  $\square$

Notice that for each fixed  $\varepsilon \in E$ , the set of functions  $f_{\varepsilon^*}(\cdot - \alpha)$ ,  $\alpha \in \mathbb{Z}^2$ , can be interpreted as being derived from  $\delta_\alpha$  by refining with the subdivision scheme  $S_\varepsilon$ . Since  $S_\varepsilon$  is interpolatory, this set of functions is linearly independent.

Some of the scaling functions which generate  $V_0$  are plotted in Figure 6. The different orientations due to the application of the adaptive directional subdivision scheme to the Dirac delta  $\delta_0$  is evident. This fact forces the associated shearlet spaces to also comprise directionality, hence to react to directional behavior of the data.

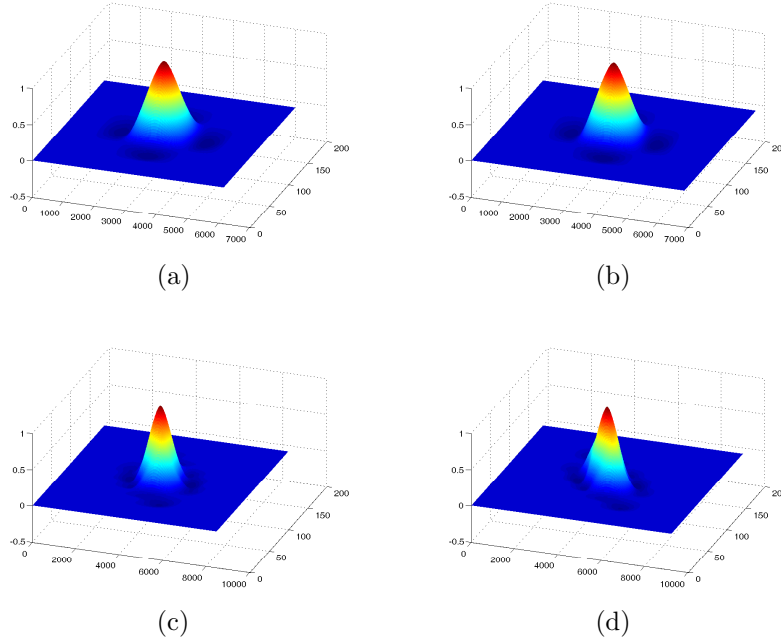


FIGURE 6. This figure shows the refinement of  $\delta_0$  after applying  $S_\varepsilon$  with (a)  $\varepsilon = (0, 0, 0, 0, 0)$ , (b)  $\varepsilon = (0, 0, 0, 1, 0)$ , (c)  $\varepsilon = (0, 1, 0, 0, 0)$ , and (d)  $\varepsilon = (0, 1, 1, 1, 1)$ .

## 7. FAST SHEARLET DECOMPOSITION

Let  $\mathcal{P}_n$ ,  $n \in \mathbb{N}_0$ , denote a sequence of projections from  $V_{n+1}$  to  $V_n$ , respectively, and define the *shearlet spaces* as  $H_n = (\mathcal{P}_n - I) V_{n+1}$ ,  $n \in \mathbb{N}_0$ , hence as an appropriate complement of  $V_n$  in  $V_{n+1}$ . In classical MRA,  $\mathcal{P}$  is chosen as an orthogonal projection, but following the approach from [18], we can also use interpolation as a projection, provided that the subdivision schemes were interpolatory.

**7.1. Refinable Functions.** In order to establish the shearlet decomposition, we require the following two observations.

**Lemma 7.1.** *For all  $\varepsilon \in E$  and  $c \in \ell(\mathbb{Z}^2)$ , we have*

$$\sum_{\alpha \in \mathbb{Z}^2} c(W_\varepsilon^{-1}\alpha) f_0(W_\varepsilon \cdot -\alpha) = \sum_{\alpha \in \mathbb{Z}^2} c(W_0^{-n}\alpha) f_0(U_\varepsilon(W_0^n \cdot -\alpha)).$$

*Proof.* Since all the matrices  $U_\varepsilon$ ,  $\varepsilon \in E$ , are unimodular, we obtain

$$\begin{aligned} \sum_{\alpha \in \mathbb{Z}^2} c(W_\varepsilon^{-1}\alpha) f_0(W_\varepsilon \cdot -\alpha) &= \sum_{\alpha \in \mathbb{Z}^2} c(W_0^{-n}U_\varepsilon^{-1}\alpha) f_0(W_\varepsilon \cdot -\alpha) \\ &= \sum_{\alpha \in \mathbb{Z}^2} c(W_0^{-n}\alpha) f_0(U_\varepsilon(U_\varepsilon^{-1}W_\varepsilon \cdot -\alpha)) \\ &= \sum_{\alpha \in \mathbb{Z}^2} c(W_0^{-n}\alpha) f_0(U_\varepsilon(W_0^n \cdot -\alpha)). \quad \square \end{aligned}$$

To formulate the next result, we denote by  $r : E \rightarrow E$  the *reversal* operator for sequences, which maps  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$  to  $r(\varepsilon) := r(\varepsilon_1, \dots, \varepsilon_n) := (\varepsilon_n, \dots, \varepsilon_1)$ . Moreover, we will write  $0_k = P_k 0^*$  for the zero sequence in  $E_k$ ,  $k \in \mathbb{N}$ . We can now derive the following crucial relationship between refinable functions and subdivision.

**Lemma 7.2.** *For  $0 \leq k \leq n$ ,  $\varepsilon = (\eta, \tau) \in E$ ,  $\eta \in E_k$  and  $c \in \ell(\mathbb{Z}^2)$ , we have*

$$\sum_{\alpha \in \mathbb{Z}^2} c(\alpha) f_{\varepsilon^*}(W_0^{n-k} \cdot -\alpha) = \sum_{\alpha \in \mathbb{Z}^2} S_\eta c(\alpha) f_{\widehat{\tau}^*}(W_{r(\eta)} W_0^{n-k} \cdot -\alpha).$$

*Proof.* Without loss of generality we can assume that  $\tau = (0)$ . Then, for  $\varepsilon = (\varepsilon_1, \widehat{\varepsilon})$ , the refinement equation (4.3) gives

$$\begin{aligned} &\sum_{\alpha \in \mathbb{Z}^2} c(\alpha) f_{\varepsilon^*}(W_0^{n-k} \cdot -\alpha) \\ &= \sum_{\alpha \in \mathbb{Z}^2} c(\alpha) \sum_{\beta \in \mathbb{Z}^2} a_{\varepsilon_1}(\beta) f_{\widehat{\varepsilon}^*}(W_{\varepsilon_1}(W_0^{n-k} \cdot -\alpha) - \beta) \\ &= \sum_{\alpha, \beta \in \mathbb{Z}^2} a_{\varepsilon_1}(\beta - W_{\varepsilon_1}\alpha) c(\alpha) f_{\widehat{\varepsilon}^*}(W_{r(\varepsilon_1, 0_{n-k})} \cdot -\beta) \\ &= \sum_{\beta \in \mathbb{Z}^2} (S_{\varepsilon_1} c)(\beta) f_{\widehat{\varepsilon}^*}(W_{r(\varepsilon_1, 0_{n-k})} \cdot -\beta) \end{aligned}$$

This is the initial step for the inductive proof that for  $j \leq k$  we have

$$(7.1) \quad \begin{aligned} &\sum_{\alpha \in \mathbb{Z}^2} c(\alpha) f_{\varepsilon^*}(W_0^{n-k} \cdot -\alpha) \\ &= \sum_{\beta \in \mathbb{Z}^2} S_{(\varepsilon_1, \dots, \varepsilon_j)} c(\beta) f_{(\varepsilon_{j+1}, \dots, \varepsilon_k)^*}(W_{r(\varepsilon_1, \dots, \varepsilon_j, 0_{n-k})} \cdot -\beta). \end{aligned}$$

Indeed, applying the refinement equation (4.3) once more to (7.1), we get that

$$\begin{aligned}
& \sum_{\alpha \in \mathbb{Z}^2} c(\alpha) f_{\varepsilon^*} (W_0^{n-k} \cdot -\alpha) \\
&= \sum_{\alpha, \beta \in \mathbb{Z}^2} S_{(\varepsilon_1, \dots, \varepsilon_j)} c(\beta) a_{\varepsilon_{j+1}}(\alpha) f_{(\varepsilon_{j+2}, \dots, \varepsilon_k)^*} (W_{\varepsilon_{j+1}} (W_{r(\varepsilon_1, \dots, \varepsilon_j, 0_{n-k})} \cdot -\beta) - \alpha) \\
&= \sum_{\alpha, \beta \in \mathbb{Z}^2} a_{\varepsilon_{j+1}}(\alpha - W_{\varepsilon_{j+1}} \beta) S_{(\varepsilon_1, \dots, \varepsilon_j)} c(\beta) f_{(\varepsilon_{j+2}, \dots, \varepsilon_k)^*} (W_{r(\varepsilon_1, \dots, \varepsilon_{j+1}, 0_{n-k})} \cdot -\alpha)
\end{aligned}$$

which advances the induction hypothesis in (7.1). Specifically, for  $j = k$  this identity gives

$$\begin{aligned}
\sum_{\alpha \in \mathbb{Z}^2} c(\alpha) f_{\varepsilon^*} (W_0^{n-k} \cdot -\alpha) &= \sum_{\beta \in \mathbb{Z}^2} S_{\varepsilon} c(\beta) f_0 (W_{r(\varepsilon, 0_{n-k})} \cdot -\beta) \\
&= \sum_{\beta \in \mathbb{Z}^2} S_{\varepsilon} c(\beta) f_0 (U_{r(\varepsilon, 0_{n-k})} W_0^n \cdot -\beta) \\
&= \sum_{\beta \in \mathbb{Z}^2} S_{\varepsilon} c(U_{r(\varepsilon, 0_{n-k})} \beta) f_0 (U_{r(\varepsilon, 0_{n-k})} (W_0^n \cdot -\beta)).
\end{aligned}$$

Since for any  $\eta \in E_k$

$$\begin{aligned}
-2^{n+1} [r(\eta, 0_{n-k})]_2 &= -2^{n+1} 2^{-n+k} \sum_{j=1}^k \eta_{k-j} 2^{-j} \\
&= -2^{k+1} \sum_{j=1}^k r(\eta)_j 2^{-j} \\
&= -2^{k+1} [r(\eta)]_2,
\end{aligned}$$

we finally get the identity

$$\begin{aligned}
\sum_{\alpha \in \mathbb{Z}^2} c(\alpha) f_{\varepsilon^*} (W_0^{n-k} \cdot -\alpha) &= \sum_{\beta \in \mathbb{Z}^2} S_{\varepsilon} c(U_{r(\varepsilon)} \beta) f_0 (U_{r(\varepsilon)} (W_0^n \cdot -\beta)) \\
&= \sum_{\alpha \in \mathbb{Z}^2} S_{\varepsilon} c(\alpha) f_0 (W_{r(\varepsilon)} W_0^{n-k} \cdot -\alpha),
\end{aligned}$$

which proves the claim.  $\square$

Now suppose we are given some data from a finely sampled function on the grid  $W_0^{-n} \mathbb{Z} = 4^{-n} \mathbb{Z} \times 2^{-n} \mathbb{Z}$ , say. The key idea for the decomposition of this data, dependent on different directions, is stated in the following result which is the backbone of the MRA based fast discrete shearlet decomposition. We would like to mention that it relies on the fact that the masks  $a_0$  and  $a_1$  are chosen to be interpolatory and thus give us an explicit expression for  $\mathcal{P}_n - I$ .

The wavelet part of such a decomposition is, as usual, related to the representatives of the quotient groups  $\Gamma_{\varepsilon} := \mathbb{Z}^2 / W_{r(\varepsilon)} \mathbb{Z}^2$ ,  $\varepsilon \in E$ . Since for  $\varepsilon \in E_n$  we have  $\det W_0 = \det W_1 = 8^n$ , all such quotient groups consist of a number of elements that depends only on the length of  $\varepsilon$ ; we will denote by  $\Gamma_{\varepsilon}^*$  a selection of  $8^n - 1$  representatives for  $\Gamma_{\varepsilon} \setminus \{[0]\}$ . In the sequel, we will make use of the notation  $D_M c = c(M \cdot)$ ,  $M$  being some  $2 \times 2$ -matrix.

**Theorem 7.3.** For  $c \in \ell(\mathbb{Z}^2)$ ,  $\varepsilon = (\eta, \tau) \in E$ ,  $\eta \in E_k$  and  $n \geq k$  we have that

$$(7.2) \quad \begin{aligned} & \sum_{\alpha \in \mathbb{Z}^2} c(\alpha) f_{\tau^*} (W_{r(\eta)} W_0^{n-k} \cdot -\alpha) = \sum_{\alpha \in \mathbb{Z}^2} c(W_{r(\eta)} \alpha) f_{\varepsilon^*} (W_0^{n-k} \cdot -\alpha) \\ & + \sum_{\gamma \in \Gamma_\eta^*} \sum_{\alpha \in \mathbb{Z}^2} (c - S_\eta D_{W_{r(\eta)}} c) (W_{r(\eta)} \alpha + \gamma) f_{\tau^*} (W_{r(\eta)} (W_0^{n-k} \cdot -\alpha) - \gamma). \end{aligned}$$

*Proof.* The decomposition is based on the prediction–correction method which has become standard for interpolation based wavelet decomposition, in particular in connection with the so-called “lazy wavelet” and the associated “lifting schemes” [31].

We subsample the data  $c \in \ell(\mathbb{Z}^2)$  to obtain  $c' = D_{W_{r(\eta)}} c$  and make use of Lemma 7.2 to obtain that

$$\sum_{\alpha \in \mathbb{Z}^2} c'(\alpha) f_{\varepsilon^*} (W_0^{n-k} \cdot -\alpha) = \sum_{\alpha \in \mathbb{Z}^2} S_\eta c'(\alpha) f_{\hat{\tau}^*} (W_{r(\eta)} W_0^{n-k} \cdot -\alpha).$$

This identity is then decomposed with respect to  $\Gamma_\eta$  giving the *prediction*

$$\begin{aligned} & \sum_{\alpha \in \mathbb{Z}^2} c'(\alpha) f_{\varepsilon^*} (W_0^{n-k} \cdot -\alpha) \\ &= \sum_{\gamma \in \Gamma_\eta} \sum_{\alpha \in \mathbb{Z}^2} S_\eta c' (W_{r(\eta)} \alpha + \gamma) f_{\tau^*} (W_{r(\eta)} W_0^{n-k} \cdot -W_{r(\eta)} \alpha - \gamma) \\ &= \sum_{\alpha \in \mathbb{Z}^2} S_\eta c' (W_{r(\eta)} \alpha) f_{\hat{\tau}^*} (W_{r(\eta)} W_0^{n-k} \cdot -W_{r(\eta)} \alpha) \\ & \quad + \sum_{\gamma \in \Gamma_\eta^*} \sum_{\alpha \in \mathbb{Z}^2} S_\eta c' (W_{r(\eta)} \alpha + \gamma) f_{\tau^*} (W_{r(\eta)} W_0^{n-k} \cdot -W_{r(\eta)} \alpha - \gamma) \\ &= \sum_{\alpha \in \mathbb{Z}^2} c(\alpha) f_{\tau^*} (W_{r(\eta)} (W_0^{n-k} \cdot -\alpha)) \\ & \quad + \sum_{\gamma \in \Gamma_\eta^*} \sum_{\alpha \in \mathbb{Z}^2} S_\eta c' (W_{r(\eta)} \alpha + \gamma) f_{\tau^*} (W_{r(\eta)} (W_0^{n-k} \cdot -\alpha) - \gamma) \end{aligned}$$

since the subdivision schemes were supposed to be interpolatory. Comparing this with the decomposition

$$\begin{aligned} \sum_{\alpha \in \mathbb{Z}^2} c(\alpha) f_{\tau^*} (W_{r(\eta)} W_0^{n-k} \cdot -\alpha) &= \sum_{\gamma \in \Gamma_\eta} \sum_{\alpha \in \mathbb{Z}^2} c(\alpha) f_{\tau^*} (W_{r(\eta)} W_0^{n-k} \cdot -W_{r(\eta)} \alpha - \gamma) \\ &= \sum_{\alpha \in \mathbb{Z}^2} c(\alpha) f_{\tau^*} (W_{r(\eta)} (W_0^{n-k} \cdot -\alpha)) \\ & \quad + \sum_{\gamma \in \Gamma_\eta^*} \sum_{\alpha \in \mathbb{Z}^2} c(\alpha) f_{\tau^*} (W_{r(\eta)} (W_0^{n-k} \cdot -\alpha) - \gamma) \end{aligned}$$

we have to apply precisely the *correction* from (7.2).  $\square$

For the special case  $\eta = \varepsilon_1$  and thus  $\tau = \hat{\varepsilon}$ , Theorem 7.3 simplifies into the following form.

**Corollary 7.4.** For  $c \in \ell(\mathbb{Z}^2)$ ,  $\varepsilon \in E$  and  $n \in \mathbb{N}$  we have that

$$(7.3) \quad \sum_{\alpha \in \mathbb{Z}^2} c(\alpha) f_{\varepsilon^*}(W_{\varepsilon_1} W_0^{n-1} \cdot -\alpha) = \sum_{\alpha \in \mathbb{Z}^2} c(W_{\varepsilon_1} \alpha) f_{\varepsilon^*}(W_0^{n-1} \cdot -\alpha) \\ + \sum_{\gamma \in \Gamma_{\varepsilon_1}^*} \sum_{\alpha \in \mathbb{Z}^2} (c - S_{\varepsilon_1} D_{W_{\varepsilon_1}} c)(W_{\varepsilon_1} \alpha + \gamma) f_{\varepsilon^*}(W_{\varepsilon_1}(W_0^{n-1} \cdot -\alpha) - \gamma).$$

*Remark 7.5.* The decomposition (7.3) is the shearlet decomposition associated with the shearlet MRA: The function on the left hand side belongs to  $V_n$  and is written as the sum of a function in  $V_{n-1}$  and correction terms from  $V_n$  that vanish at  $W_{\varepsilon_1} \mathbb{Z}^2$  – the *shearlets* in the interpolatory MRA.

**7.2. Decomposition Algorithm.** The *fast shearlet decomposition* is now based on an iterative application of (7.3), where each step can be understood as filtering by means of a filter bank. To that end, we have to interpret the initial sequence  $c \in \ell(\mathbb{Z}^2)$  appropriately. Denoting by  $g_\varepsilon := f_0(U_\varepsilon \cdot)$  the “sheared” version of the refinable function  $f_0$ , we form the quasi-interpolants

$$(7.4) \quad q_{\varepsilon,n} := g_\varepsilon * (D_{U_\varepsilon} c)(W_0^n \cdot) = \sum_{\alpha \in \mathbb{Z}^2} c(U_\varepsilon \alpha) g_\varepsilon(W_0^n \cdot -\alpha) = \sum_{\alpha \in \mathbb{Z}^2} c(\alpha) f_0(U_\varepsilon W_0^n \cdot -\alpha).$$

These are precisely the functions which appear on the left hand side of (7.2) and (7.3). It is worthwhile to note that all the functions  $q_{\varepsilon,n}$  are relying on the same initial data  $c \in \ell(\mathbb{Z}^2)$ .

The interpretation of (7.4) is rather easy now if we take into account that  $f_0$  was assumed to be the limit function of an interpolatory scheme, hence cardinal:  $f_0(\alpha) = \delta_{0,\alpha}$ ,  $\alpha \in \mathbb{Z}^2$ . Hence, since

$$(7.5) \quad q_{\varepsilon,n}(x) = \sum_{\alpha \in \mathbb{Z}^2} c(\alpha) f_0(W_\varepsilon x - \alpha), \quad x \in \mathbb{R}^2,$$

we can substitute  $x = W_\varepsilon^{-1} \alpha = M_\varepsilon \alpha$  and use the cardinality of  $f_0$  to find that  $q_{\varepsilon,n}(M_\varepsilon \alpha) = c(\alpha)$  or  $q_{\varepsilon,n}(W_0^{-n} \alpha) = c(U_\varepsilon \alpha)$ , respectively. The latter tells us that we should interpret the sequence  $c$  as a function sampled at the grid  $W_0^{-n} \mathbb{Z}^2$ , while the parameter  $\varepsilon$  determines how this data is sheared and which thus are the directions “preferred” by the wavelet decomposition.

For the fast decomposition we now start with  $c \in \ell(\mathbb{Z}^2)$ , interpret it as in (7.5), and decompose it in two ways, namely, for  $\varepsilon \in E_1$ , into

$$q_{\varepsilon,n} = \sum_{\alpha \in \mathbb{Z}^2} c_\varepsilon(\alpha) f_{\varepsilon^*}(W_0^{n-1} \cdot -\alpha) + \sum_{\gamma \in \Gamma_\varepsilon^*} \sum_{\alpha \in \mathbb{Z}^2} d_{\varepsilon,\gamma}(\alpha) f_0(W_\varepsilon(W_0^{n-1} \cdot -\alpha) - \gamma),$$

where the coefficients

$$c_\varepsilon = D_{W_\varepsilon} c \\ d_{\varepsilon,\gamma} = (c - S_\varepsilon D_{W_\varepsilon} c)(W_\varepsilon \cdot + \gamma)$$

are obtained by filtering the original sequence  $c$  in *both* cases. This is the fundamental property of this decomposition algorithm: even if we decompose *two different* functions,  $q_{\varepsilon,n}$  with  $\varepsilon \in E_1$ , we have to filter *only one* data vector to obtain the new set of scaling coefficients  $\{c_\varepsilon : \varepsilon \in E_1\}$  and shearlet coefficients  $\{d_{\varepsilon,\gamma} : \varepsilon \in E_1, \gamma \in \Gamma_\varepsilon^*\}$ .

In the next step, the sequences  $c_\varepsilon$  and the associated functions  $q_{(\varepsilon,\eta),n-1}$  are decomposed in precisely the same way, making use of Corollary 7.4 again. Like above, we filter  $c_0$  twice to obtain new, further downsampled sequences  $c_{(0,0)}$  and  $c_{(0,1)}$  together with the respective shearlet coefficients  $d_{(0,0),\gamma}$ ,  $\gamma \in \Gamma_0^*$  and  $d_{(0,1),\gamma}$ ,  $\gamma \in \Gamma_1^*$ . In exactly the same way we obtain  $c_{(1,0)}$  and  $c_{(1,1)}$  as well as  $d_{(1,0),\gamma}$ ,  $\gamma \in \Gamma_0^*$  and  $d_{(1,1),\gamma}$ ,  $\gamma \in \Gamma_1^*$  by filtering  $c_1$ . These first two steps of decomposition are illustrated in Figure 7. It can already be seen from Figure 7 that – like the

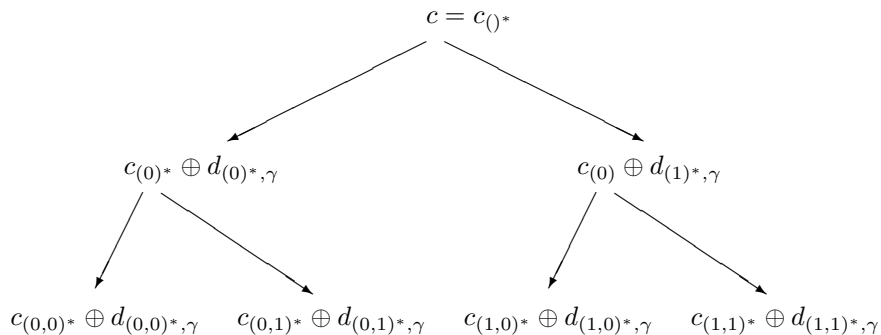


FIGURE 7. The binary tree associated with the fast shearlet decomposition.

subdivision scheme – the shearlet decomposition becomes a binary tree labeled by the directional indices  $\varepsilon$ . Indeed, in general we obtain the new coefficients by the following simple filtering.

**Algorithm 7.6.** *Let  $c_\varepsilon$  for some  $\varepsilon \in E$  be given. Then the next level of scaling and shearlet coefficients are computed as*

$$\begin{aligned} c_{(\varepsilon,\eta)} &= D_{W_\eta} c_\varepsilon, \\ d_{(\varepsilon,\eta),\gamma} &= (c_\varepsilon - S_\eta D_{W_\eta} c_\varepsilon)(W_\eta \cdot + \gamma), \quad \eta \in E_1, \quad \gamma \in \Gamma_\eta^*. \end{aligned}$$

Eventually, this process ends up with coarsest level scaling coefficients  $c_\varepsilon$ ,  $\varepsilon \in E_n$ , and shearlet coefficients  $d_{\varepsilon,\gamma}$ ,  $\varepsilon \in E_k$ ,  $k \leq n$ ,  $\gamma \in \Gamma_{\varepsilon_k}^*$  which describe the deviation from the coarse data.

Indeed, it is now easily seen that such a decomposition must recognize “sheared” and thus directional components of two dimensional data since (7.2) relates, for  $\varepsilon \in E$ , the data  $D_{W_\varepsilon} c$  with the function  $g_\varepsilon$  and the respective shearlet coefficients must be large where the prediction by the subdivision scheme is inaccurate, i.e., at directional singularities. Thus, the “recipe” is to consider the shearlet coefficients

$$d_{P_k \varepsilon, \gamma}, \quad k = 1, \dots, n, \quad \varepsilon \in E_n, \quad \gamma \in \Gamma_{\varepsilon_k}^*.$$

A precise analysis of this nevertheless fundamental aspect of directional edge detection is beyond the scope of this paper where we just want to give the framework for adaptive directional detections. It should also be clear that the adaptive directional approach is not tied to interpolatory schemes, in fact, any perfect reconstruction filter bank can be used as long as the projection and its complement can be expressed properly. We plan to address these questions as well as the numerical implementations in a further paper, however.

## ACKNOWLEDGMENTS

The first author would like to thank Ingrid Daubechies for very inspiring discussions, and Wolfgang Dahmen for helpful comments on an earlier version of this paper. She especially thanks PACM at Princeton University for its hospitality and support during her visit.

## REFERENCES

1. J-P. Antoine, R. Murenzi, and P. Vandergheynst, Directional wavelets revisited: Cauchy wavelets and symmetry detection in patterns, *Appl. Comput. Harmon. Anal.* **5** (1999), 314–345.
2. E. J. Candès and L. Demanet, *The curvelet representation of wave propagators is optimally sparse*, *Comm. Pure Appl. Math.* **58** (2005), 1472–1528.
3. E. J. Candès and D. L. Donoho, *Ridgelets: a key to higher-dimensional intermittency?*, *Phil. Trans. R. Soc. Lond. A.* **357** (1999), 2495–2509.
4. ———, *New tight frames of curvelets and optimal representations of objects with  $C^2$  singularities*, *Comm. Pure Appl. Math.* **56** (2004), 219–266.
5. ———, *Continuous curvelet transform: I. Resolution of the wavefront set*, *Appl. Comput. Harmon. Anal.* **19** (2005), 162–197.
6. A. S. Cavaretta, W. Dahmen, and C. A. Micchelli, *Stationary Subdivision*, *Mem. Amer. Math. Soc.* **93** 1991.
7. M. Charina and C. Conti, *Convergence of multivariate non-stationary vector subdivision schemes*, *Appl. Numer. Math.* **49** (2004), 343–354.
8. M. Charina, C. Conti, and T. Sauer,  *$L_p$ -convergence of subdivision schemes: joint spectral radius versus restricted spectral radius*, in *Approximation theory XI* (Gatlinburg, TN, 2004), M. Neamtu and L. L. Schumaker, eds., Nashboro Press, Nashville, TN (2005), 129–150.
9. A. Cohen and N. Dyn, *Nonstationary subdivision schemes and multiresolution analysis*, *SIAM J. Math. Anal.* **27** (1996), 1745–1769.
10. D. Cox, J. Little, and D. O’Shea, *Ideals, Varieties and Algorithms*, Undergraduate Texts in Mathematics, Springer–Verlag, 1992.
11. S. Dahlke, G. Kutyniok, P. Maass, C. Sagiv, H.-G. Stark, and G. Teschke, *The Uncertainty Principle Associated with the Continuous Shearlet Transform*, *Int. J. Wavelets Multiresolut. Inf. Process.*, to appear.
12. S. Dahlke, G. Kutyniok, G. Steidl, and G. Teschke, *Shearlet Coorbit Spaces and associated Banach Frames*, preprint (2007).
13. W. Dahmen and C. A. Micchelli, *Biorthogonal wavelet expansion*, *Constr. Approx.* **13** (1997), 294–328.
14. I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia (1992).
15. G. Deslauriers and S. Dubuc, *Symmetric iterative interpolation processes*, *Constr. Approx.* **5** (1989), 49–68.
16. M. N. Do and M. Vetterli, *The contourlet transform: an efficient directional multiresolution image representation*, *IEEE Trans. Image Process.* **14** (2005), 2091–2106.
17. D. L. Donoho, *Interpolating wavelet transforms*, Technical Report, Department of Statistics, Stanford University, 1992.
18. G. Faber, *Über stetige Funktionen*, *Math. Ann.* **66** (1909), 81–94.
19. K. Guo, G. Kutyniok, and D. Labate, *Sparse Multidimensional Representations using Anisotropic Dilation und Shear Operators*, in *Wavelets und Splines* (Athens, GA, 2005), G. Chen und M. J. Lai, eds., Nashboro Press, Nashville, TN (2006), 189–201.
20. K. Guo and D. Labate, *Optimally sparse multidimensional representations using shearlets*, *SIAM J. Math. Anal.* **39** (2007), 298–318.
21. N. Kingsbury, *Complex wavelets for shift invariant analysis and filtering of signals*, *Appl. Comput. Harmon. Anal.* **10**(3) (2001), 234–253.
22. G. Kutyniok and D. Labate, *Resolution of the Wavefront Set using Continuous Shearlets*, preprint (2006).
23. G. Kutyniok and T. Sauer, *From Wavelets to Shearlets and back again*, in *Approximation Theory XII* (San Antonio, TX, 2007), C. K. Chui, M. Neamtu, and L. Schumaker, eds., Nashboro Press, Nashville, TN, to appear.

24. D. Labate, W-Q. Lim, G. Kutyniok, and G. Weiss, *Sparse multidimensional representation using shearlets*, in Wavelets XI (San Diego, CA, 2005), M. Papadakis, A. F. Laine und M. A. Unser, eds., SPIE Proc. 5914, SPIE, Bellingham, WA (2005), 254–262.
25. H. M. Möller and T. Sauer, *H-bases for polynomial interpolation and system solving*, Adv. Comp. Math. **12** (2000), 335–362.
26. ———, *Multivariate refinable functions of high approximation order via quotient ideals of Laurent polynomials*, Adv. Comp. Math. **20** (2004), 205–228.
27. T. Sauer, *Gröbner bases, H-bases and interpolation*, Trans. Amer. Math. Soc. **353** (2001), 2293–2308.
28. ———, *Polynomial interpolation, ideals and approximation order of refinable functions*, Proc. Amer. Math. Soc. **130** (2002), 3335–3347.
29. ———, *Stationary vector subdivision – quotient ideals, differences and approximation power*, Rev. R. Acad. Cien. Serie A. Mat. **96** (2002), 257–277.
30. ———, *Differentiability of multivariate refinable functions and factorization*, Adv. Comp. Math. **25** (2006), 211–235.
31. W. Sweldens, *The lifting scheme: a custom-design construction of biorthogonal wavelets*, Appl. Comput. Harmon. Anal. **3** (1996), 186–200.

PROGRAM IN APPLIED AND COMPUTATIONAL MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544, USA

*E-mail address:* `kutyniok@math.princeton.edu`

INSTITUTE OF MATHEMATICS, JUSTUS-LIEBIG-UNIVERSITY GIESSEN, 35392 GIESSEN, GERMANY

*E-mail address:* `tomas.sauer@math.uni-giessen.de`