On Holomorphic Artin L-functions

Florin Nicolae
Technische Universität Berlin
Institut für Mathematik MA 8-1
Strasse des 17. Juni 136
D-10623 Berlin
nicolae@math.tu-berlin.de
and
Institute of Mathematics of the Romanian Academy
P.O.BOX 1-764
RO-014700 Bucharest

March 22, 2013

Abstract

Let $K/\mathbb{Q}$ be a finite Galois extension, $s_0 \in \mathbb{C} \setminus \{1\}$, $Hol(s_0)$ the semigroup of Artin L-functions holomorphic at s_0. We present criteria for Artin's holomorphy conjecture in terms of the semigroup $Hol(s_0)$. We conjecture that Artin's L-functions are holomorphic at s_0 if and only if $Hol(s_0)$ is factorial. We prove this if s_0 is a zero of an L-function associated to a linear character of the Galois group.

Key words: Artin L-function; Artin's holomorphy conjecture
MSC: 11R42

1. Introduction

Let $K/\mathbb{Q}$ be a finite Galois extension with the Galois group G, $\chi_1, \ldots, \chi_r$ the irreducible characters of G with the dimensions $d_1 := \chi_1(1), \ldots, d_r := \chi_r(1)$, $f_1 = L(s, \chi_1, K/\mathbb{Q}), \ldots, f_r = L(s, \chi_r, K/\mathbb{Q})$ the correponding L-functions, $Ar := \{f_1^{k_1} \cdot \ldots \cdot f_r^{k_r} \mid k_1 \geq 0, \ldots, k_r \geq 0\}$ the multiplicative semigroup with the basis $f_1, \ldots, f_r$. (Artin proved that $f_1, \ldots, f_r$ are multiplicatively independent [1], Satz 5, P. 106 .) For $s_0 \in \mathbb{C}, s_0 \neq 1$ let $Hol(s_0)$ be the subsemigroup of Ar consisting of the L-functions which are holomorphic at s_0. Artin's conjecture is:

$$Hol(s_0) = Ar.$$

For $f, g \in Ar$ we write $g \mid f$ if there exists $h \in Ar$ such that $f = gh$.

Theorem 1. *The following assertions are equivalent:*
1) Artin's conjecture is true: $Hol(s_0) = Ar$.
2) The semigroup $Hol(s_0)$ *is factorial and*

$$\prod_{j\in J} f_j \in Hol(s_0)$$

for every subset $J \subset \{1, \ldots, r\}$ *with* $r - 1$ *elements.*
3) The semigroup $Hol(s_0)$ *is factorial and for any* $k \in \{1, \ldots, r\}$ *there exists* $f \in Hol(s_0)$ *such that*

$$f_k \nmid f \text{ in } Ar \text{ and } f_l \mid f \text{ in } Ar, \forall l \neq k.$$

4) The semigroup $Hol(s_0)$ *is factorial and for any* $k, l \in \{1, \ldots, r\}$, $k \neq l$ *there exists* $f \in Hol(s_0)$ *such that*

$$f_k \nmid f \text{ in } Ar \text{ and } f_l \mid f \text{ in } Ar.$$

We have seen in [4], p. 2862, that $Hol(s_0)$ is a positive affine semigroup which generates the free abelian group with the basis $f_1, \ldots, f_r$. The Hilbert basis of $Hol(s_0)$ is the uniquely determined minimal system of generators of $Hol(s_0)$. It follows that $Hol(s_0)$ is factorial if and only if the number of elements in its Hilbert basis is r. So the assertion 1) $\Leftrightarrow$ 2) is Theorem 2 from [4].

By the results of Heilbronn [3], Stark [5], Theorem 3, p. 144, and Foote-Murty [2] it is known that the poles of Artin's L-functions are zeros of the Dedekind zeta function $\zeta_K(s)$. It holds that

$$\zeta_K(s) = f_1^{d_1} \cdot \ldots \cdot f_r^{d_r}, \tag{1}$$

so it is sufficient to test Artin's conjecture at a point s_0 which is a zero of some f_k, $k \in \{1, \ldots, r\}$. If G is isomorphic to A_5, the alternating group on five elements, it was proved in [4], Theorem 3, that the following assertions are equivalent:
1) Artin's L-functions are holomorphic at s_0.
2) The semigroup $Hol(s_0)$ *is factorial.*
We conjecture that this holds for any Galois group G:
Conjecture. *Let* $K/\mathbb{Q}$ *be a finite Galois extension. The following assertions*

are equivalent:
1) Artin's L-functions are holomorphic at s_0 *.*
2) The semigroup $Hol(s_0)$ *is factorial.*
We prove the conjecture in the following special case:

Theorem 2. *If* χ *is a linear character of* G*, i.e.* $\chi(1) = 1$*, and* $L(s_0, \chi) = 0$*, then the conjecture is true.*

2. Proofs

Proof of Theorem 1:
1)⇒2) is trivial because Ar is factorial.
2)⇒3): For $k \in \{1, \ldots, r\}$ and $f := \prod_{\substack{j=1 \\ j \neq k}}^{r} f_j$ it holds that $f \in Hol(s_0)$ and

$$f_k \nmid f \text{ in } Ar, f_l \mid f \text{ in } Ar, \forall l \neq k.$$

3)⇒4) is trivial.
4)⇒1) For a meromorphic function f we denote by $\mathrm{ord}_{s=s_0} f$ its order at s_0. If Artin's conjecture is false, then there exists $k \in \{1, \ldots, r\}$ such that $\mathrm{ord}_{s=s_0} f_k < 0$. From (1) it follows that

$$\mathrm{ord}_{s=s_0} \zeta_K = \sum_{j=1}^{r} d_j \mathrm{ord}_{s=s_0} f_j \geq 0, \tag{2}$$

so there exists $j \neq k$ such that

$$\mathrm{ord}_{s=s_0} f_j > 0.$$

For $l \in \{1, \ldots, r\}$ let

$$m_l := \min\{m \geq 0 \mid \mathrm{ord}_{s=s_0} f_j^{m_l} \cdot f_l \geq 0\}.$$

Since $\mathrm{ord}_{s=s_0} f_k < 0$, we have that $m_k > 0$. The elements $f_j^{m_1} \cdot f_1, \ldots, f_j^{m_r} \cdot f_r$ are irreducible in $Hol(s_0)$. Since, by the assumption 4), the positive affine semigroup $Hol(s_0)$ ([4], Remark 8) is factorial and generates a free abelian group of rank r ([4], p. 2862), its Hilbert basis $\mathrm{Hilb}(Hol(s_0))$, the minimal system of generators, consists of r elements. It follows that

$$\mathrm{Hilb}(Hol(s_0)) = \{f_j^{m_1} \cdot f_1, \ldots, f_j^{m_r} \cdot f_r\}.$$

By assumption 4) there exists $f \in Hol(s_0)$ such that

$$f_j \nmid f \text{ in } Ar \text{ and } f_k \mid f \text{ in } Ar.$$

The element $f \in Hol(s_0)$ is a product of elements from $\text{Hilb}(Hol(s_0))$, hence there exist $n_1, \ldots, n_r \geq 0$ such that

$$f = (f_j^{m_1} f_1)^{n_1} \cdot \ldots \cdot (f_j^{m_r} f_r)^{n_r},$$

so

$$f_j^{m_k n_k} \mid f.$$

Since $f_k \mid f$, we have that $n_k > 0$. Since $m_k > 0$, it follows that $m_k n_k \geq 1$, hence $f_j \mid f$, a contradiction. So Artin's conjecture is true. □

Proof of Theorem 2:

1) ⇒ 2): If Artin's L-functions are holomorphic at s_0, then $Hol(s_0) = Ar$ is factorial.

2) ⇒ 1): Suppose that there exists $k \in \{1, \ldots, r\}$ such that $\text{ord}_{s=s_0} f_k < 0$. By renumbering we take $L(s, \chi) = f_1$ and $k = 2$, so $d_1 = 1$, $\text{ord}_{s=s_0} f_1 > 0$, and $\text{ord}_{s=s_0} f_2 < 0$. For $j \in \{1, \ldots, r\}$ let

$$m_j := \min\{m \geq 0 \,|\} \text{ord}_{s=s_0}(f_1^m \cdot f_j) \geq 0.$$

Since $\text{ord}_{s=s_0} f_2 < 0$ it follows that $m_2 > 0$. The elements $f_1, f_1^{m_2} \cdot f_2, \ldots, f_1^{m_r} \cdot f_r$ are irreducible in $Hol(s_0)$. Since by assumption 2) $Hol(s_0)$ is factorial its Hilbert basis consists of r elements. It follows that

$$\text{Hilb}(Hol(s_0)) = \{f_1, f_1^{m_2} \cdot f_2, \ldots, f_1^{m_r} \cdot f_r\},$$

hence

$$Hol(s_0) = \{f_1^{k_1} \cdot (f_1^{m_2} \cdot f_2)^{k_2} \cdot \ldots \cdot (f_1^{m_r} \cdot f_r)^{k_r} \mid k_1, k_2, \ldots, k_r \geq 0\} =$$
$$= \{f_1^{k_1 + m_2 k_2 + \ldots + m_r k_r} \cdot f_2^{k_2} \cdot \ldots \cdot f_r^{k_r} \mid k_1, k_2, \ldots, k_r \geq 0\}.$$

Since $\zeta_K \in Hol(s_0)$ there exist $k_1, \ldots, k_r \geq 0$ such that

$$\zeta_K = f_1^{k_1 + m_2 k_2 + \ldots + m_r k_r} \cdot f_2^{k_2} \cdot \ldots \cdot f_r^{k_r}.$$

From (1) it follows that

$$1 = d_1 = k_1 + m_2 k_2 + \ldots + m_r k_r$$

and

$$d_2 = k_2,$$

hence

$$m_2 d_2 \leq 1,$$
$$m_2 = d_2 = 1$$

since $m_2 > 0$ and $m_2 > 0$. So the Artin L-function f_2 is associated to a character of degree $d_2 = 1$. By class field theory this function is a Hecke L-function, so it is holomorphic at s_0, a contradiction. □

References

[1] Artin, E.: Über eine neue Art von L-Reihen. Abh. Math. Sem. Hamburg 3, 89-108 (1924). 2

[2] Foote, R; Murty, V.Kumar: Zeros and poles of Artin L-series. Math. Proc. Camb. Philos. Soc. 1, 5-11 (1989).

[3] Heilbronn, H.: On real zeros of Dedekind -functions. Can. J. Math. 25, 870-873 (1973). 2

[4] Nicolae, F.: On the semigroup of Artin's L-functions holomorphic at s_0. J. Number Theory 128, No. 11, 2861-2864 (2008). 2

[5] Stark, H.M.: Some Effective Cases of the Brauer-Siegel Theorem. Invent. Math. 23, 135-152(1974). 2, 3

2