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Error estimates for parabolic optimal control problems with control constraints

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An optimal control problem for the 1-d heat equation is investigated with pointwise control constraints. This paper is concerned with the discretization of the control by piecewise linear functions. The connection between the solutions of the discretized problems and the continuous one is investigated. Under an additional assumption on the adjoint state an approximation order $\sigma^{3/2}$ is proved for uniform discretizations. In the general case it is shown that a non-uniform control discretization ensure an approximation of order $\sigma^{3/2}$. Numerical tests confirm the theoretical part.
AMS subject classification: 49N10, 49M05, 65K10

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Abstract

An optimal control problem for the 1-d heat equation is investigated with pointwise control constraints. This paper is concerned with the discretization of the control by piecewise linear functions. The connection between the solutions of the discretized problems and the continuous one is investigated. Under an additional assumption on the adjoint state an approximation order $\sigma^{3/2}$ is proved for uniform discretizations. In the general case it is shown that a non-uniform control discretization ensure an approximation of order $\sigma^{3/2}$. Numerical tests confirm the theoretical part.

Keywords: Linear-quadratic optimal control problems, error estimates, heat equation, non-uniform grids, numerical approximation, control constraints.

AMS subject classification: 49N10, 49M05, 65K10

1 Introduction

The paper is concerned with the discretization of parabolic optimal control problems. We discuss here the case of a boundary control of the 1-d heat equation

$$\begin{aligned} y_t(t,x) &= y_{xx}(t,x) && \text{in } (0,T)\times(0,1) \\ y(0,x) &= y^o(x) && \text{in } (0,1) \\ y_x(t,0) &= 0 && \text{in } (0,T) \\ y_x(t,1) &= u(t) && \text{in } (0,T) \end{aligned} \tag{1.1}$$

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$$(P) \begin{cases} \min \quad J(u) = \frac{1}{2}\|y(T,.) - y_d\|_Y^2 + \frac{\nu}{2}\|u\|_U^2 \\ \text{subject to (1.1)} \\ \text{and } u \in C = \{u \in U |\ a \le u \le b \quad a.e. \text{ on } [0,T]\}, \end{cases}$$

where $U = L^2(0,T)$, $Y = L^2(0,1)$, and $y^o, y_d \in Y$. Moreover, $\nu > 0$, a, b are real numbers with $a > b$. We only discuss the effect of the control discretization here. Therefore, we introduce the control discretized counterpart of (P)

$$(P_\sigma) \begin{cases} \min \quad J(u) = \frac{1}{2}\|y(T,.) - y_d\|_Y^2 + \frac{\nu}{2}\|u\|_U^2 \\ \text{subject to (1.1)} \\ \text{and } u \in C_\sigma = \{u \in U_\sigma |\ a \le u \le b \quad a.e. \text{ on } [0,T]\}. \end{cases}$$

Here, the state space Y is not discretized.

In this paper, we discuss a space U_σ of *piecewise linear functions on uniform and non-uniform grids.*

Elliptic optimal control problems discretized by piecewise constant functions are well investigated, we refer to Falk [3], Geveci [4], and Arada/Casas/Tröltzsch [1]. The authors show for different examples the convergence order σ. The parabolic case with controls piecewise constant in time is discussed in Malanowski [12]. In that paper a convergence rate $\sigma^{1/2}$ is proved for parabolic optimal control problems with boundary control.

Two difficulties occur in our simple problem (P) which are typical for parabolic optimal control problems. First, the optimal control does not belong to the space $H^1(0,T)$ in general. Therefore it is not possible to apply the results of Casas/Tröltzsch [2] or Rösch [13] directly. Hence, we cannot expect approximation order σ or higher.

In [13] it is shown that in sufficiently regular cases the convergence rate $\sigma^{3/2}$ is obtained. We will describe situations where this approximation order for uniform grids holds in this paper. Moreover, the convergence rate $\sigma^{3/2}$ is proved for suitable non-uniform grids in a general case.

A completely different way is gone by Hinze [7]. In that approach only the state space is discretized. The optimal control can be obtained by projection of the adjoint state to the set of admissible controls. Therefore, the obtained optimal controls do not belong to finite dimensional subspaces of U.

Linear-quadratic optimal control problems are subproblems of SQP-algorithms and other higher order methods solving nonlinear optimal control problems, see for instance Heinkenschloss/Tröltzsch [6], Kelley/Sachs [8], Kunisch/Sachs [10], Tröltzsch [14] and the references therein.

The linear-quadratic optimal control problems can be attacked by a primal-dual active set strategy, see Hager [5] or Kunisch/Rösch [9]. The undiscretized optimal control problem can be solved theoretically using this strategy with an arbitrary high accuracy. Unfortunately, we are not able to solve the appearing system of equations exactly. Thus, it is necessary to discretize control and state. Therefore the approximation error of the solution of the discretized problem with respect to the continuous one plays an important role.

We describe the discretized space U_h in the usual form: For a given grid $t_i \in [0,T]$ $(i = 0..n)$ with $t_0 = 0$ and $t_n = T$ we define the functions e_i by

$$e_i = \begin{cases} \dfrac{x - t_{i-1}}{t_i - t_{i-1}} & \text{if } x \in [t_{i-1}, t_i) \\[2ex] \dfrac{t_{i+1} - x}{t_{i+1} - t_i} & \text{if } x \in [t_i, t_{i+1}] \\[2ex] 0 & \text{otherwise.} \end{cases}$$

Now $u \in U_\sigma$ can be expressed as

$$u = \sum_{i=0}^{n} u_i e_i.$$

Moreover, $u \in C_\sigma$ means

$$a \le u_i \le b \quad \forall i = 0..n.$$

Thus, (P_σ) can be equivalently written as

$$(P'_\sigma) \begin{cases} \min \quad J(u) = \frac{1}{2}\|y - y_d\|_Y^2 + \frac{\nu}{2}\|u\|_U^2 \\ \text{subject to } y = Su \\ u = \sum\limits_{i=0}^{n} u_i e_i \\ \text{and } a \le u_i \le b \quad \forall i = 0..n \end{cases}$$

in this case.

The paper is organized as follows: In section 2 we formulate the optimality conditions and state the main results. Section 3 contains several auxiliary results. The proof of the error estimates is presented in section 4. The paper ends with numerical tests in section 5.

2 Optimality conditions

First, we introduce the spaces $V = H^1(0,1)$ and $W(0,T) = \{v \in L^2(0,T;V) : v_t \in L^2(0,T;V^*)\}$. A weak solution $y \in W(0,T)$ is defined by the equations

$$\begin{aligned} \int_0^1 y_t \cdot v \, dx + \int_0^1 y_x \cdot v_x \, dx &= u \cdot v(1) \\ y(0) &= y^o \end{aligned}$$

for almost all t and all $v \in V$. For the theory of weak solutions in $W(0,T)$, we refer to Lions/Magenes [11]

Lemma 1 *For each $u \in L^2(0,T)$, equation (1.1) admits a unique solution $y \in W(0,T)$. Hence, $y(T,.)$ belongs to Y.*

We introduce now the adjoint equation

$$\begin{aligned} -p_t(t,x) &= p_{xx}(t,x) && \text{in } (0,T)\times(0,1) \\ p(T,x) &= y(T,x) - y_d(x) && \text{in } (0,1) \\ p_x(t,0) &= 0 && \text{in } (0,T) \\ p_x(t,1) &= 0 && \text{in } (0,T). \end{aligned} \tag{2.1}$$

Lemma 2 *For each $y \in W(0,T)$, the adjoint equation (2.1) admits a unique solution $p \in W(0,T)$. Hence, $p(.,1)$ belongs to U.*

Now we are able to formulate the necessary and sufficient optimality conditions for (P) and (P_σ).

Lemma 3 *Let $\bar{u} \in C$ an admissible control for (P) with associated state $\bar{y}$ and adjoint state $\bar{p}$ defined by (1.1) and (2.1). Then $\bar{u}$ is the optimal solution of (P) if and only if*

$$(\bar{p}(.,1) + \nu\bar{u}, u - \bar{u})_U \geq 0 \tag{2.2}$$

holds for all $u \in C$. Analogue, let $u_\sigma \in C_h$ be an admissible control for (P_σ) with associated state $\bar{y}$ and adjoint state $\bar{p}$ defined by (1.1) and (2.1). Then u_σ is optimal for (P_σ) if and only if

$$(p_\sigma(.,1) + \nu u_\sigma, u - u_\sigma)_U \geq 0 \tag{2.3}$$

is fulfilled for all $u \in C_\sigma$.

The regularity of the adjoint state $\bar{p}$ plays an important role for error estimates. This regularity is influenced by the desired state y_d. Therefore, we require here an additional regularity assumption for y_d.

Assumption (A): The desired state y_d belongs to $H^2(0,1)$.

Now we are able to state the main results of the paper:

Theorem 1 *Suppose that*

$$-\frac{\bar{p}(T,1)}{\nu} \notin [a,b] \tag{2.4}$$

and assumption (A) holds. Moreover, $[0,T]$ is discretized uniformly ($t_i = i \cdot T/n$). Then the estimate

$$\|\bar{u} - u_\sigma\|_U \le c \cdot \sigma^{3/2} \tag{2.5}$$

is fulfilled for the optimal solutions $\bar{u}$ of (P) and u_σ of (P_σ) with a positive constant $c > 0$ and $\sigma = T/n$.

Theorem 2 *Suppose that*

$$-\frac{\bar{p}(T,1)}{\nu} \neq a, \qquad -\frac{\bar{p}(T,1)}{\nu} \neq b, \tag{2.6}$$

and assumption (A) holds. Moreover, $[0,T]$ is discretized in a suitable non-uniform way (for instance $t_i = T - T \cdot \frac{(n-i)^4}{n^4}$ $(i = 0..n)$). Then the estimate

$$\|\bar{u} - u_\sigma\|_U \le c \cdot \sigma^{3/2} \tag{2.7}$$

is fulfilled for the optimal solutions $\bar{u}$ of (P) and u_σ of (P_σ) with a positive constant $c > 0$ and $\sigma = T/n$.

The proofs of Theorem 1 and Theorem 2 are contained in section 4.

3 Auxiliary results

First, we introduce the Green's function

$$G(t,x,\xi) = \sum_{n=0}^{\infty} v_n(x) v_n(\xi) e^{-n^2\pi^2 t}, \tag{3.1}$$

where $v_n(x) = \sqrt{2}\cos n\pi x$ denote the normalized eigenfunctions of a Sturm–Liouville eigenvalue problem associated with the problem (1.1): $v_{xx} = \lambda v,\ v_x = 0$ at $x = 0,\ x = 1$.

For further investigations we need some estimates concerning the infinite sum in this function:

Lemma 4 *For $0 < t \le T$ the estimates*

$$\sum_{n=0}^{\infty} e^{-n^2\pi^2 t} \le ct^{-1/2} \tag{3.2}$$

$$\sum_{n=0}^{\infty} n^{-2}(1 - e^{-n^2\pi^2 t}) \le ct^{1/2} \tag{3.3}$$

$$\sum_{n=0}^{\infty} n^2 e^{-n^2\pi^2 t} \le ct^{-3/2} \tag{3.4}$$

$$\sum_{n=0}^{\infty} n^4 e^{-n^2\pi^2 t} \le ct^{-5/2} \tag{3.5}$$

hold true.

Proof: We use a generic constant $c > 0$

$$\begin{aligned}\sum_{n=0}^{\infty} e^{-n^2\pi^2 t} &\leq 1 + \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \\ &\leq 1 + \int_0^{\infty} e^{-x^2\pi^2 t}\, dx \\ &\leq 1 + c \cdot t^{-1/2} \\ &\leq c \cdot t^{-1/2}.\end{aligned}$$

For the last inequality we used that $c \cdot t^{-1/2} > 1$ on $(0, T]$ is fulfilled for a sufficiently large constant c. Therefore, estimate (3.2) holds true. Integrating this inequality, we obtain

$$\int_0^s \sum_{n=0}^{\infty} e^{-n^2\pi^2 t}\, dt \leq c \cdot s^{-1/2}.$$

Changing integration and summation, we get

$$\sum_{n=0}^{\infty} (\pi n)^{-2}(1 - e^{-n^2\pi^2 s}) \leq c s^{1/2}.$$

The last two inequalities can be proved by similar arguments. □

By the Fourier method, we get for the solution of (1.1)

$$y(t,x) = \int_0^t G(t-s, x, 1) u(s)\, ds + \int_0^1 G(t, x, \xi) y^o(\xi)\, d\xi. \tag{3.6}$$

Using Green's function for the adjoint equation (2.1), we obtain the formula

$$p(t,x) = \int_0^1 G(T-t, x, \xi)(y(T,\xi) - y_d(\xi))\, d\xi$$

and in particular for the boundary values of the adjoint state yields

$$p(t,1) = \int_0^1 G(T-t, 1, \xi)(y(T,\xi) - y_d(\xi))\, d\xi. \tag{3.7}$$

The boundary values of the optimal adjoint state $\bar{p}$ are directly coupled with the optimal control $\bar{u}$ via the formula

$$\bar{u} = \Pi_{[a,b]}(-\frac{1}{\nu}\bar{p}(.,1)) \tag{3.8}$$

with the projection operator

$$\Pi_{[a,b]}(r) = \begin{cases} a & \text{for } r < a \\ r & \text{for } r \in [a,b] \\ b & \text{for } r > b. \end{cases} \tag{3.9}$$

This is a consequence of the optimality condition (2.2).

Next, we discuss the regularity of the optimal control and the optimal adjoint state.

Lemma 5 *The optimal control $\bar{u}$ belongs to the space $C^{0,1/2}[0,T]$.*

Proof: The proof is mainly standard. We sketch here only the main ideas. Inserting (3.6) in (3.7), we obtain

$$\begin{aligned}
\bar{p}(t,1) &= \int_0^1\int_0^T G(T-t,1,\xi)G(T-s,\xi,1)\bar{u}(s)\,ds\,d\xi \\
&\quad + \int_0^1\int_0^1 G(T-t,1,\xi)G(T,\xi,\zeta)y^o(\zeta)\,d\zeta\,d\xi - \int_0^1 G(T-t,1,\xi)y_d(\xi)\,d\xi \\
&= \int_0^T \sum_{n=0}^{\infty} e^{-n^2\pi^2(2T-t-s)}\bar{u}(s)\,ds + \int_0^1 \sum_{n=0}^{\infty} e^{-n^2\pi^2(2T-t)}v_n(\zeta)y^o(\zeta)\,d\zeta \\
&\quad - \int_0^1 \sum_{n=0}^{\infty} e^{-n^2\pi^2(T-t)}v_n(\xi)y_d(\xi)\,d\xi
\end{aligned}$$

Hence, $\bar{p}(.,1)$ is analytic in every interval $[0,\alpha]$ with $\alpha < T$. For the regularity we investigate the crucial point $t = T$. We obtain for the difference $\bar{p}(T,1) - \bar{p}(t,1)$

$$\begin{aligned}
\bar{p}(T,1) - \bar{p}(t,1) &= \int_0^T \sum_{n=0}^{\infty}(e^{-n^2\pi^2(T-s)} - e^{-n^2\pi^2(2T-t-s)})\bar{u}(s)\,ds \\
&\quad + \int_0^1 \sum_{n=0}^{\infty}(e^{-n^2\pi^2T} - e^{-n^2\pi^2(2T-t)})v_n(\zeta)y^o(\zeta)\,d\zeta \\
&\quad - \int_0^1 \sum_{n=0}^{\infty}(1 - e^{-n^2\pi^2(T-t)})v_n(\xi)y_d(\xi)\,d\xi
\end{aligned}$$

The second term contains no singularity for $t \nearrow T$. Because of $y_d \in H^2(0,1)$ we have

$$\left|\int_0^1 v_n(\xi)y_d(\xi)\,d\xi\right| \le \frac{c}{n^2}.$$

From this estimate and (3.3), we get

$$\left| \int_0^1 \sum_{n=0}^{\infty} (1 - e^{-n^2\pi^2(T-t)}) v_n(\xi) y_d(\xi)\, d\xi \right| \leq c\sqrt{T-t}.$$

It remains the first term. Using the positivity of each addend, we obtain

$$\left| \int_0^T \sum_{n=0}^{\infty} (e^{-n^2\pi^2(T-s)} - e^{-n^2\pi^2(2T-t-s)}) \bar{u}(s) ds \right| \leq c_m \int_0^T \sum_{n=0}^{\infty} (e^{-n^2\pi^2(T-s)} - e^{-n^2\pi^2(2T-t-s)}) ds$$

with $c_m = \max(|a|, |b|)$. We continue by

$$\begin{aligned}
\left| \int_0^T \sum_{n=0}^{\infty} e^{-n^2\pi^2(T-s)} - e^{-n^2\pi^2(2T-t-s)}) \bar{u}(s)\, ds \right| &\leq c_m \int_0^T \sum_{n=0}^{\infty} e^{-n^2\pi^2(T-s)}\, ds \\
&\quad - c_m \int_{t-T}^{t} \sum_{n=0}^{\infty} e^{-n^2\pi^2(T-s)}\, ds \\
&= -c_m \int_{t-T}^{0} \sum_{n=0}^{\infty} e^{-n^2\pi^2(T-s)}\, ds \\
&\quad + c_m \int_t^T \sum_{n=0}^{\infty} e^{-n^2\pi^2(T-s)}\, ds \\
&\leq 0 + c_m \int_t^T \frac{c}{\sqrt{T-s}}\, ds \\
&\leq c\sqrt{T-t}
\end{aligned}$$

using inequality (3.2) in the last step. Combining the results, we end up with

$$|\bar{p}(T,1) - \bar{p}(t,1)| \leq c\sqrt{T-t},$$

that implies

$$|\bar{u}(T) - \bar{u}(t)| \leq c\sqrt{T-t}$$

because of (3.8). $\square$

Remark 1 *The result of Lemma 5 can also be proved for free or unilateral constrained parabolic optimal control problems. In this case, it has to be proved the boundedness of the optimal control in a first step.*

Remark 2 *Compatibility conditions have to be fulfilled for higher regularity of $\bar{p}(.,1)$, especially*

$$(\bar{y}(T,.) - y_d)_x(1) = 0.$$

We can not expect this property in general.

Remark 3 *It is possible to prove that $\bar{p}(.,1)$ belongs to the Sobolev space $H^{1-\varepsilon}(0,T)$ for $\varepsilon > 0$ using similar arguments like in the proof of Lemma 5. Again, the compatibility condition is needed for higher regularity.*

The properties of $\bar{p}(.,1)$ are transferred to $\bar{u}$ by the projection formula (3.8). Clearly, $\bar{u}$ is smoother than $\bar{p}(.,1)$ if the projection cut the singular behaviour in a neighbourhood of T. This will be the key point in the proof of Theorem 1 in the next section. The situation is much more complicate if the projection is simply the identity. For this purpose, we formulate the next lemma.

Lemma 6 *The function $\bar{p}(t,1)$ is twice differentiable with respect to t in $[0,T)$. Moreover, it holds*

$$|\bar{p}_{tt}(t,1)| \le c|T-t|^{-3/2}. \tag{3.10}$$

Proof: We start again with the formula

$$\begin{aligned}\bar{p}(t,1) &= \int_0^T \sum_{n=0}^{\infty} e^{-n^2\pi^2(2T-t-s)}\bar{u}(s)\,ds + \int_0^1 \sum_{n=0}^{\infty} e^{-n^2\pi^2(2T-t)} v_n(\zeta) y^o(\zeta)\,d\zeta \\ &\quad - \int_0^1 \sum_{n=0}^{\infty} e^{-n^2\pi^2(T-t)} v_n(\xi) y_d(\xi)\,d\xi.\end{aligned}$$

Of course, $p(t,1)$ is analytic in every interval $[0,\alpha]$ with $\alpha < T$. We differentiate twice ($t < T$) and obtain

$$\begin{aligned}\bar{p}_{tt}(t,1) &= \int_0^T \sum_{n=0}^{\infty} n^4\pi^4 e^{-n^2\pi^2(2T-t-s)}\bar{u}(s)\,ds + \int_0^1 \sum_{n=0}^{\infty} n^4\pi^4 e^{-n^2\pi^2(2T-t)} v_n(\zeta) y^o(\zeta)\,d\zeta \\ &\quad - \int_0^1 \sum_{n=0}^{\infty} n^4\pi^4 e^{-n^2\pi^2(T-t)} v_n(\xi) y_d(\xi)\,d\xi.\end{aligned}$$

To get estimate (3.10), we have to estimate the absolute values of these three integrals. The first integral can be estimated using (3.5) since u is pointwise bounded. The second integral has no singularity at T. It remains the last integral. Assumption (A) implies

$$|\int_0^1 v_n(\xi) y_d(\xi)\,d\xi| \le c \cdot n^2.$$

Therefore we can use (3.4) to estimate the third integral. Consequently, we obtain

$$|\bar{p}_{tt}(t,1)| \le c \cdot \int_0^T (2T-t-s)^{-5/2}\,dt + c + c\cdot(T-t)^{-3/2} \le c|T-t|^{-3/2}$$

and the assertion is proved. $\square$

Clearly, estimate (3.10) is transferred to the optimal control $\bar{u}$ excepted the non-differentiability points of the projection term. We are now able to prove a first approximation result.

Lemma 7 *Let the discretization $t_i = T - T \cdot \frac{(n-i)^4}{n^4}$ $(i = 0..n)$ be given. Furthermore, let $v = \sum_{i=0}^{n} \bar{p}(t_i)e_i$ be the linear interpolate of $\bar{p}$. Then, the estimate*

$$\|v - \bar{p}(.,1)\|_U \leq \frac{c}{n^2} \tag{3.11}$$

holds true.

Proof: Let $i < n$. Then, in every interval $[t_{i-1}, t_i]$ we have

$$\bar{p}(t_{i-1}, 1) = v(t_{i-1}) \quad \text{and} \quad \bar{p}(t_i, 1) = v(t_i).$$

From this, we find easily

$$|\bar{p}(t,1) - v(t)| \leq c \max_{s \in [t_{i-1}, t_i]} |\bar{p}_{tt}(s,1)||t_i - t_{i-1}|^2 \qquad \forall t \in [t_{i-1}, t_i]$$

Using (3.10), we obtain

$$|\bar{p}(t,1) - v(t)| \leq c(T - t_i)^{-3/2}|t_i - t_{i-1}|^2 \qquad \forall t \in [t_{i-1}, t_i].$$

Therefore, we can estimate the integral

$$\int_{t_{i-1}}^{t_i} (\bar{p}(t,1) - v(t))^2 \, dt \leq c(T - t_i)^{-3}|t_i - t_{i-1}|^4 \, dt \leq c(T - t_i)^{-3}|t_i - t_{i-1}|^5.$$

We continue by

$$\begin{aligned} \|v - \bar{p}(.,1)\|_U^2 &= \int_0^T (\bar{p}(t,1) - v(t))^2 \, dt \\ &\leq \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} (\bar{p}(t,1) - v(t))^2 \, dt + \int_{t_{n-1}}^{t_n} (\bar{p}(t,1) - v(t))^2 \, dt. \end{aligned} \tag{3.12}$$

Because of the continuity of $\bar{p}$ we can estimate the second integral by

$$\int_{t_{n-1}}^{t_n} (\bar{p}(t,1) - v(t))^2 \, dt \leq c(t_n - t_{n-1}) = \frac{c}{n^4}. \tag{3.13}$$

It remains to discuss the sum in (3.12).

$$\begin{aligned} \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} (\bar{p}(t,1) - v(t))^2 \, dt &\leq c \cdot \sum_{i=1}^{n-1} (T - t_i)^{-3}|t_i - t_{i-1}|^5 \\ &\leq c \cdot \sum_{i=1}^{n-1} \left(T \left(\frac{n-(i-1)}{n} \right)^4 - T \left(\frac{n-i}{n} \right)^4 \right)^5 \left(T \cdot \left(\frac{n-i}{n} \right)^4 \right)^{-3} \end{aligned}$$

Changing the summation order we obtain

$$
\begin{aligned}
\sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} (\bar{p}(t,1) - v(t))^2 \, dt &\leq c \cdot \sum_{i=1}^{n-1} \left(T \left(\frac{i+1}{n} \right)^4 - T \left(\frac{i}{n} \right)^4 \right)^5 \left(T \cdot \left(\frac{i}{n} \right)^4 \right)^{-3} \\
&\leq c \cdot \sum_{i=1}^{n-1} \left(T \frac{4i^3 + 6i^2 + 4i + 1}{n^4} \right)^5 \left(T \cdot \left(\frac{i}{n} \right)^4 \right)^{-3} \\
&\leq c \cdot \sum_{i=1}^{n-1} \frac{1}{n^8} T^5 (4+6+4+1)^5 i^{15} \cdot i^{-12} T^{-3} \\
&\leq c \cdot \frac{1}{n^4} \qquad (3.14)
\end{aligned}
$$

Inserting (3.14) and (3.13) in (3.12), we end up with

$$
\|v - \bar{p}(.,1)\|_U^2 \leq c \cdot \frac{1}{n^4}
$$

that implies the assertion. □

4 Error estimates

First, we recall a result of Casas/Tröltzsch [2]. For this purpose we define a function via

$$
v_\sigma^i = \begin{cases} a & \text{if } \min_{t \in [t_{i-1}, t_{i+1}]} \bar{u}(t) = a \\ b & \text{if } \max_{t \in [t_{i-1}, t_{i+1}]} \bar{u}(t) = b \\ \bar{u}(t_i) & else \end{cases}
$$

and

$$
v_\sigma = \sum_{i=0}^{n} v_\sigma^i e_i.
$$

If the mesh size σ is sufficiently small, then $\bar{u}(t) = a$ and $\bar{u}(t') = b$ can not happen in the same interval $[t_{i-1}, t_{i+1}]$.

Lemma 8 *(Casas/Tröltzsch [2]) The function v_σ fulfil the condition*

$$
(\bar{p} + \nu \bar{u}, v - v_\sigma)_U \geq 0 \qquad (4.1)
$$

for all $v \in C_\sigma$. Moreover, it holds

$$
\|\bar{u} - u_\sigma\|_U \leq c \|\bar{u} - v_\sigma\|_U . \qquad (4.2)
$$

For the proof we refer to [2] Lemma 2.1 and the discussion afterwards.

Now, we are able to prove the main results presented in section 2.

Proof of Theorem 1: Here we have a uniform discretization and $\sigma = \frac{1}{n}$. We start with condition (2.4). Because of the continuity of $\bar{p}$, there exists an interval $[t', T]$

$$-\frac{\bar{p}(t,1)}{\nu} \notin [a,b] \qquad \forall t \in [t', T].$$

From (3.8), we know that this interval belongs to the set of active constraints and we have $\bar{u} \equiv a$ or $\bar{u} \equiv b$ on $[t', T]$. On the other interval $[0, t']$ the adjoint state $\bar{p}(.,1)$ is analytic.

Next, we estimate the term $\|\bar{u} - v_\sigma\|_U$, where v_σ is the function defined above. For that purpose we subdivide the intervals $[t_{i-1}, t_i]$ in three disjoint classes

$$\begin{aligned} I_1 &:= \{i \in \{1,..,n\} : \; v_\sigma(t_{i-1}), v_\sigma(t_i) \in (a,b)\}, \\ I_2 &:= \{i \in \{1,..,n\} : \; \bar{u}(t) \equiv a \text{ or } \bar{u}(t) \equiv b \text{ on } [t_{i-1}, t_i]\}, \\ I_3 &:= [0,T] \setminus (I_1 \cup I_2). \end{aligned}$$

We continue by

$$\|\bar{u} - v_\sigma\|_U^2 = \|\bar{u} - v_\sigma\|_{L^2(I_1)}^2 + \|\bar{u} - v_\sigma\|_{L^2(I_2)}^2 + \|\bar{u} - v_\sigma\|_{L^2(I_3)}^2.$$

By definition we obtain

$$\|\bar{u} - v_\sigma\|_{L^2(I_2)}^2 = 0.$$

On I_1, the function v_σ is the linear interpolate of $\bar{u}$. Moreover, on this set $\bar{u}$ is analytic and belongs therefore especially to C^2. Hence we obtain

$$\|\bar{u} - v_\sigma\|_{L^2(I_1)}^2 \leq c \cdot \sigma^4.$$

It remains to estimate the last norm. The set I_3 can intersect the interval $[t', T]$. For sufficiently fine discretization, it is possible to find a t'' with $t' \leq t'' < T$ such that $I_3 \cap [t'', T] = \emptyset$. Since $\bar{p}(.,1)$ is analytic on $[0, t'']$, there exist only finitely many points with $-\frac{1}{\nu}\bar{p}(t,1) = a$ or $-\frac{1}{\nu}\bar{p}(t,1) = b$ on $[0, t']$. Therefore, the number of intervals $[t_{i-1}, t_i]$ which are subsets of I_3 can be bounded by a finite number K independent of the discretization. The optimal control $\bar{u}$ is lipschitz continuous on I_3 and we can estimate

$$\|\bar{u} - v_\sigma\|_{L^2(I_3)}^2 = \int_{I_3} (\bar{u} - v_\sigma)^2 \, dt \leq cKL^2 \cdot \sigma^3.$$

From this, we get

$$\|\bar{u} - v_\sigma\|_U \leq c \cdot \sigma^{3/2}$$

and formula (4.2) implies the assertion. □

In this proof, we benefit from the active control part at the end of the interval $[0, T]$. If inequality (2.4) is not valid, then the projection formula (3.8) does not cut the problematic part at the end of the time interval. It may happen that the set I_1 contains an interval $[t^*, T]$. From section 3, we know that $\bar{u}$ is in general only $C^{1/2}[0, T]$ (Lemma 5) or $H^{1-\varepsilon}(0, T)$. Applying the same technique as in the proof before, we obtain:

Remark 4 *Let assumption (A) be fulfilled and let inequality (2.4) not be valid. Then an error estimate*

$$\|\bar{u} - u_\sigma\|_U \leq c \cdot \sigma^{1-\varepsilon}$$

is obtained with $\varepsilon > 0$ for uniform grids.

In this case a non-uniform grid can improve the convergence rate. In the proof of Theorem 2 we combine the technique of the last proof with the result of Lemma 7.

Proof of Theorem 2: First, we find for the non-uniform grid $t_i = T - T \cdot \frac{(n-i)^4}{n^4}$ $(i = 0..n)$:

$$\sigma = \max_{i=1..n} |t_i - t_{i-1}| = t_1 - t_0 \leq \frac{4}{n}.$$

It is easy to see that $\sigma \geq \frac{3}{n}$ for sufficiently large n. We have to discuss two cases. In the first case, inequality (2.4) is fulfilled. Here, we can directly apply the arguments of the proof of Theorem 1 to obtain the assertion. In the second case, we have

$$-\frac{1}{\nu}\bar{p}(T, 1) \in (a, b).$$

Consequently, there exists an interval $[t^*, T]$ with

$$-\frac{1}{\nu}\bar{p}(t, 1) \in (a, b) \qquad \text{for all } t \in [t^*, T].$$

Without lost of generality, let t^* be a grid point. Next, we change the definition of the sets I_1, I_2, I_3 a little bit. We use the same definition but replace T by t^*. On this interval we can apply the argumentation of the proof of Theorem 1 and obtain

$$\|\bar{u} - v_\sigma\|^2_{L^2(0,t^*)} \leq c \cdot \sigma^3.$$

For the interval $[t^*, T]$ we find

$$\|\bar{u} - v_\sigma\|^2_{L^2(t^*,T)} = \left\| -\frac{1}{\nu}\bar{p}(., 1) + \frac{1}{\nu} v \right\|^2_{L^2(t^*,T)},$$

where v is the linear interpolate of $\bar{p}(., 1)$ defined in Lemma 7. Now, Lemma 7 implies

$$\|\bar{u} - v_\sigma\|^2_{L^2(t^*,T)} \leq \frac{1}{\nu^2}\|\bar{p}(., 1) - v\|^2_U \leq \frac{c}{n^4} \leq c \cdot \sigma^4.$$

Therefore, we have

$$\|\bar{u} - v_\sigma\|_U \leq c \cdot \sigma^{3/2}.$$

and again the assertion follows from inequality (4.2). $\square$

5 Numerical Tests

In this section we present some illustrating numerical tests confirming the theoretical results. The linear-quadratic optimal control problems were solved by a primal-dual active set strategy, see Hager [5] or Kunisch/Rösch [9]. The parabolic equations are discretized by a Crank-Nicolson scheme. The space discretization in all numerical examples is $n_x = 250$.

Example 1: This example is constructed in such a way that no constraint is active.

$$T = 1.58, \quad \nu = 0.01, \quad y_d = \frac{1}{2}(1 - x^2), \quad a = -10, \quad b = 10$$

First, we present the uniform discretization. We compare all solutions with $\hat{u} = u_{1024}$. The last column contains the value

$$\text{rate} := \frac{\ln \|u_n - \hat{u}\| - \ln \|u_{512} - \hat{u}\|}{\ln 512 - \ln n} \tag{5.1}$$

which is an approximation of the convergence rate.

n	$\|\|u_n - \hat{u}\|\|_U$	$n \cdot \|\|u_n - \hat{u}\|\|_U$	$n^2 \cdot \|\|u_n - \hat{u}\|\|_U$	rate
8	0.0687	0.5496	4.3966	1.01482
16	0.0670	1.0727	17.1632	1.21075
32	0.0346	1.1067	35.4138	1.27469
64	0.0115	0.7359	47.0956	1.17001
128	0.0026	0.3270	95.3775	0.66998
256	0.0015	0.3726	300.2653	0.52810
512	0.0010	0.5167	264.5663	–

The convergence is at most linear. The next table shows the numerical result for the non-uniform grid.

n	$\|\|u_n - \hat{u}\|\|_U$	$n \cdot \|\|u_n - \hat{u}\|\|_U$	$n^2 \cdot \|\|u_n - \hat{u}\|\|_U$	rate
8	0.052232	0.4179	3.3429	2.06228
16	0.013005	0.2081	3.3292	2.07355
32	0.003242	0.1037	3.3199	2.09093
64	0.000808	0.0517	3.3094	2.11972
128	0.000200	0.0256	3.2743	2.17188
256	0.000048	0.0122	3.1356	2.28133
512	0.000010	0.0050	2.5801	–

The convergence rate is here quadratic. This numerical result confirms the theoretical results due to the fact $I_3 = \emptyset$. The numerical error for the non-uniform grid for 64 intervals is smaller than for the uniform grid with 512 intervals.

The solutions for $n = 16$ and $n = 32$ are plotted in figure 1. These pictures show the problems of the uniform grids to fit the solution part at the end of the time interval.

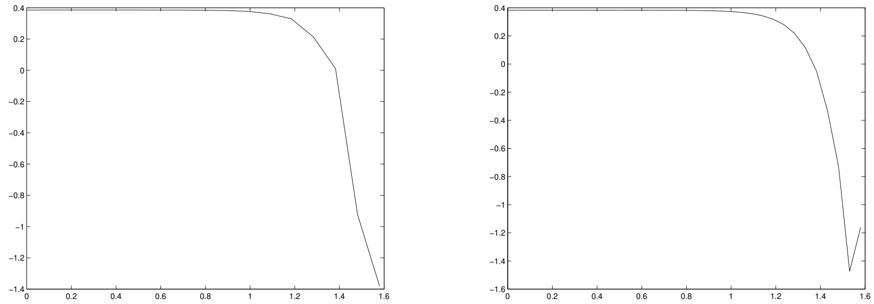

Figure 1: Uniform grids n=16 and n=32

Figure 2 illustrate the adapted behaviour of the non-uniform mesh. The grid $n = 16$ (left picture) describes the solution part at the end of the time interval in a sufficient manner. The right picture shows the solution for $n = 1024$.

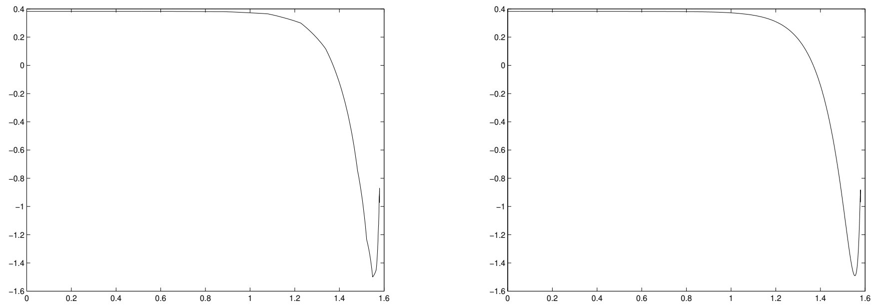

Figure 2: Non-uniform grid n=16 and n=1024

Example 2: This example is a real challenge for the non-uniform grid. Again, the solution oszillates heavily at the end of the time interval. The data are chosen in such a way that no constraint is active.

$$T = 1.58, \quad \nu = 0.0001, \quad y_d = \frac{1}{2}(1-x), \quad a = -10, \quad b = 10$$

The next table show the quadratic convergence rate for the non-uniform grid.

n	$\|\|u_n - \hat{u}\|\|_U$	$n \cdot \|\|u_n - \hat{u}\|\|_U$	$n^2 \cdot \|\|u_n - \hat{u}\|\|_U$	rate
8	0.248817	1.9905	15.9243	1.91795
16	0.121227	1.9396	31.0341	2.09406
32	0.027076	0.8664	27.7261	2.07692
64	0.006543	0.4187	26.7984	2.08620
128	0.001611	0.2062	26.3908	2.11824
256	0.000390	0.0999	25.5747	2.19116
512	0.000086	0.0438	22.4008	–

The optimal solution for $n = 16$ and $n = 1024$ is plotted in Figure 3.

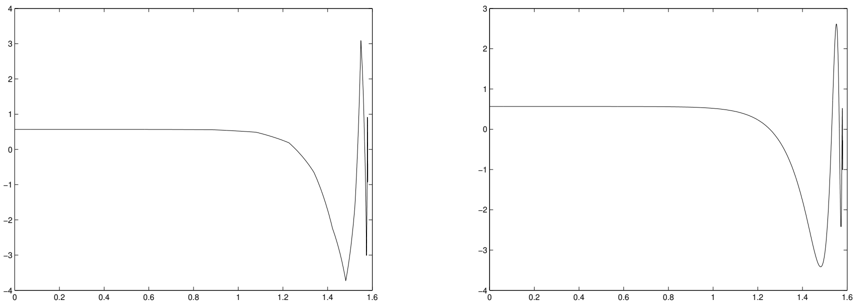

Figure 3: Non-uniform grid n=16 and n=1024

Example 3: This example illustrates the fact that active control constraints improve the convergence rate for uniform grids. The difference between the Examples 1 and 3 is the choice of a.

$$T = 1.58, \quad \nu = 0.01, \quad y_d = \frac{1}{2}(1 - x^2), \quad a = 0, \quad b = 10$$

In the next table we slightly change the presentation of the numerical results.

n	$\|\|u_n - \hat{u}\|\|_U$	$n \cdot \|\|u_n - \hat{u}\|\|_U$	$n^{3/2} \cdot \|\|u_n - \hat{u}\|\|_U$	$n^2 \cdot \|\|u_n - \hat{u}\|\|_U$
8	0.0083496	0.0668	0.1889	0.5344
16	0.0068280	0.1092	0.4370	1.7480
32	0.0040008	0.1280	0.7442	4.0968
64	0.0005978	0.0383	0.3060	2.4485
128	0.0003959	0.0507	0.5733	6.4859
256	0.0001814	0.0464	0.7431	11.8889
512	0.0000064	0.0033	0.0745	1.6845

The table shows clearly the non-uniform convergence behaviour. This effect can be easily declared. In the proof of Theorem 1, we have a quadratic convergence order on the sets I_1 and I_2. In contrast to this, the convergence order on the set I_3 is only linear. Therefore, the I_3-part influences the convergence heavily. If we refine the grid we have two cases. In the first case, the smallest distance between the corner of the optimal control and a grid point does not change or change insignificant. In the second case, a new grid point fits the corner much better than all points of the coarser grid. These two cases lead to faster and slower phases in the convergence process. In the first two examples, we calculated convergence rates. Here, we abstain from a presentation of such rates, because a change from 512 to 256 in formula (4.1) causes to large differences in these rates.

The next figure shows the optimal solution for $n = 16$ and $n = 1024$.

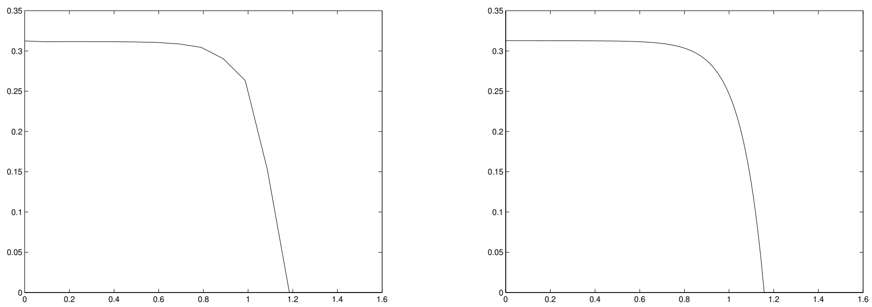

Figure 4: Non-uniform grid n=16 and n=1024

The next table shows the results for the non-uniform grid. The refinement at the end of the interval is in this case useless.

n	$\|\|u_n - \hat{u}\|\|_U$	$n \cdot \|\|u_n - \hat{u}\|\|_U$	$n^{3/2} \cdot \|\|u_n - \hat{u}\|\|_U$	$n^2 \cdot \|\|u_n - \hat{u}\|\|_U$
8	0.0482019	0.3856	1.0907	3.0849
16	0.0181645	0.2906	1.1625	4.6501
32	0.0017108	0.0547	0.3097	1.7519
64	0.0004213	0.0270	0.2157	1.7257
128	0.0002350	0.0301	0.3403	3.8504
256	0.0001613	0.0413	0.6608	10.5720
512	0.0000954	0.0488	1.1048	24.9976

The numerical solutions for $n = 16$ and $n = 32$ are plotted in the next figure. Here, the convergence rate of the non-uniform grid and the uniform grid is the same. Nevertheless, the uniform grid produces a slightly better solution for the same discretization in this case.

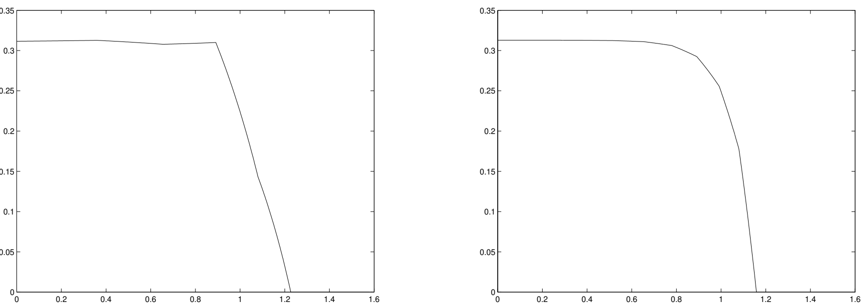

Figure 5: Non-uniform grid n=16 and n=1024

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