Lamperti transform for multi-type CBI

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(based on a joint work with J. Teichmann)

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Structure of the talk

1 Introduction
   - Real valued CB and Lamperti transform
   - Extension to real valued CBI

2 What is a multi–type CBI
   - Definitions
   - Some additional results

3 Lamperti transform for multi–type CBI
   - Multi–type CB and their time–change representation
   - Extension to multi–type CBI
   - Sketch of the proof
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Branching processes

Let

\[(\Omega, (X_t)_{t \geq 0}, (\mathcal{F}^X_t)_{t \geq 0}, (\mathbb{P}^x)_{x \in \mathbb{R} \geq 0})\]

be a time homogeneous Markov process. The process \(X\) is said to be an branching process if it satisfies the following property:

**Branching property**

For any \(t \geq 0\) and \(x_1, x_2 \in \mathbb{R} \geq 0\), the law of \(X_t\) under \(\mathbb{P}^{x_1+x_2}\) is the same as the law of \(X_t^{(1)} + X_t^{(2)}\), where each \(X^{(i)}\) has the same distribution as \(X\) under \(\mathbb{P}^{x_i}\), for \(i = 1, 2\).
Fourier–Laplace transform characterization

- There exists a function $\Psi : \mathbb{R}_{\geq 0} \times \mathbb{C}_{\leq 0} \to \mathbb{C}$ such that

$$E^X \left[ e^{uX_t} \right] = e^{x\Psi(t,u)},$$

for all $x \in \mathbb{R}_{\geq 0}$ and $(t, u) \in \mathbb{R}_{\geq 0} \times \mathbb{C}_{\leq 0}$.

- On the set $Q = \mathbb{R}_{\geq 0} \times \mathbb{C}_{\leq 0}$, the function $\Psi$ satisfies the equation

$$\partial_t \Psi(t, u) = R(\Psi(t, u)), \quad \Psi(0, u) = u.$$

The branching mechanism

The function $R$ has the following Lévy-Khintchine form

$$R(u) = \beta u + \frac{1}{2} u^2 \alpha + \int_0^\infty (e^{u\xi} - 1 - u\xi 1_{\{|\xi| \leq 1\}}) M(d\xi),$$

where $\beta \in \mathbb{R}$, $\alpha \geq 0$ and $M$ is a Lévy measure with support in $\mathbb{R}_{\geq 0}$. 
Lévy processes

A time homogeneous Markov process $Z$ is a Lévy process if the following three conditions are satisfied:

**L1)** $Z_0 = 0$ $\mathbb{P}$-a.s.

**L2)** $Z$ has independent and stationary increments, i.e. for all $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \ldots < t_{n+1} < \infty$

(indpendence) the random variables $\{Z_{t_{j+1}} - Z_{t_j}\}_{j=0,\ldots,n}$ are independent,

(stationarity) the distribution of $Z_{t_{j+1}} - Z_{t_j}$ coincides with the distribution of $Z_{(t_{j+1} - t_j)}$,

**L3)** (stochastic continuity) for each $a > 0$ and $s \geq 0$,

$$\lim_{t \to s} \mathbb{P}(|Z_t - Z_s| > a) = 0.$$
Relation with infinitely divisible distributions

- If $Z$ is a Lévy process, then, for any $t \geq 0$, the random variable $Z_t$ is infinitely divisible.

- The Fourier transform of a Lévy process takes the form:

$$
E^0 \left[ e^{\langle u, Z_t \rangle} \right] = e^{t\eta(u)}, \quad u \in \mathbb{i}\mathbb{R}
$$

where

$$
\eta(u) = \beta u + \frac{1}{2} u^2 \alpha + \int \left( e^{u\xi} - 1 - u\xi \mathbb{1}_{\{|\xi| \leq 1\}} \right) M(d\xi),
$$

where $\beta \in \mathbb{R}$, $\alpha \geq 0$ and $M$ is a Lévy measure in $\mathbb{R}$.

- The Fourier transform can be extended in the complex domain and the resulting Fourier–Laplace transform is well defined in

$$
\mathcal{U} := \{u \in \mathbb{C} \mid \eta(\Re(u)) < \infty\}.
$$
Lamperti transform

**Theorem [Lamperti, 1967]**

Let $Z$ be a Lévy process with no negative jumps with Lévy exponent $R$, i.e.

$$\mathbb{E}^0 \left[ e^{uZ_t} \right] = e^{tR(u)}, \quad u \in \mathcal{U}.$$  

Define, for $t \geq 0$

$$X_t = x + Z_{\theta_t \wedge \tau_0^-} \quad \theta_t := \inf \left\{ s > 0 \mid \int_0^s \frac{dr}{Z_r} > t \right\}.$$  

Then $X$ is a CB process with branching mechanism $R$. 
Lamperti transform

Theorem 2 in [Caballero et al., 2013]

Let $Z$ be a Lévy process with no negative jumps with Lévy exponent $R$, i.e.

$$
\mathbb{E}^0 \left[ e^{uZ_t} \right] = e^{tR(u)}, \quad u \in \mathcal{U}.
$$

The time–change equation

$$
X_t = x + Z \int_0^t x_r \, dr
$$

admits a unique solution, which is a CB process with branching mechanism $R$. 

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   - Some additional results

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   - Multi–type CB and their time–change representation
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   - Sketch of the proof
A CBI-process with branching mechanism $R$ and immigration mechanism $F$ is a Markov process $Z$ taking values in $\mathbb{R}_{\geq 0}$ satisfying

- there exist functions $\phi: \mathbb{R}_{\geq 0} \times U \to \mathbb{C}$ and $\psi: \mathbb{R}_{\geq 0} \times U \to \mathbb{C}$ such that
  \[
  \mathbb{E}^x \left[ e^{ux} \right] = e^{\phi(t,u) + u\psi(t,u)},
  \]
  for all $x \in \mathbb{R}_{\geq 0}$ and $(t,u) \in \mathbb{R}_{\geq 0} \times U$.

- On the set $Q = \mathbb{R}_{\geq 0} \times U$, the functions $\phi$ and $\psi$ satisfy the following system:
  \[
  \partial_t \phi(t,u) = F(\psi(t,u)), \quad \phi(0,u) = 0, \\
  \partial_t \psi(t,u) = R(\psi(t,u)), \quad \psi(0,u) = u.
  \]
The immigration mechanism

The function $F$ has the following Lévy-Khintchine form

$$F(u) = bu + \int_0^\infty (e^{u\xi} - 1) \, m(d\xi),$$

with $b \geq 0$ and $m$ is a Lévy measure on $\mathbb{R}_{\geq 0}$ such that

$$\int (1 \wedge \xi) m(d\xi) < \infty.$$
Theorem 2 in [Caballero et al., 2013]

Let $Z^{(1)}$ be a Lévy process with no negative jumps and $Z^{(0)}$ an independent subordinator such that

$$
\mathbb{E}^0 \left[ e^{uZ_t^{(1)}} \right] = e^{tR(u)} \quad \text{and} \quad \mathbb{E}^0 \left[ e^{uZ_t^{(0)}} \right] = e^{tF(u)}, \quad u \in \mathcal{U}.
$$

The time–change equation

$$
X_t = x + Z_t^{(0)} + \int_0^t X_r \, dr
$$

admits a unique solution, which is a CBI process with branching mechanism $R$ and immigration mechanism $F$. 

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Definition

Let

\[(\Omega, (X_t)_{t \geq 0}, (F_t^\aleph)_{t \geq 0}, (\mathbb{P}^x)_{x \in \mathbb{R}^m_{\geq 0}})\]

be a time homogeneous Markov process. The process \(X\) is said to be a multi–type \textbf{CBI} if it satisfies the following property:

See [Duffie et al., 2003, Barczy et al., 2014]

There exist functions \(\phi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}\) and \(\Psi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}^m\) such that

\[
\mathbb{E}^x \left[ e^{\langle u, X_t \rangle} \right] = e^{\phi(t,u)} + \langle x, \Psi(t,u) \rangle,
\]

for all \(x \in \mathbb{R}^m_{\geq 0}\) and \((t, u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}\), with \(\mathcal{U} = \mathbb{C}^m_{\leq 0}\).
Generalized Riccati equations

On the set $Q = \mathbb{R}_{\geq 0} \times \mathcal{U}$, the functions $\phi$ and $\psi$ satisfy the following system of generalized Riccati equations:

$$\begin{align*}
\partial_t \phi(t, u) &= F(\psi(t, u)), \quad \phi(0, u) = 0, \\
\partial_t \psi(t, u) &= R(\psi(t, u)), \quad \psi(0, u) = u.
\end{align*}$$

Lévy–Khintchine form for the vector fields

The functions $F$ and $R_k$, for each $k = 1, \ldots, m$, have the following Lévy-Khintchine form

$$\begin{align*}
F(u) &= \langle b, u \rangle + \int_{\mathbb{R}_{\geq 0} \setminus \{0\}} \left( e^{\langle u, \xi \rangle} - 1 \right) m(d\xi), \\
R_k(u) &= \langle \beta_k, u \rangle + \frac{1}{2} u_k^2 \alpha_k + \int_{\mathbb{R}_{\geq 0} \setminus \{0\}} \left( e^{\langle u, \xi \rangle} - 1 - u_k \xi_k \mathbb{1}_{\{\xi \leq 1\}} \right) M_k(d\xi).
\end{align*}$$
Admissible parameters

The set of parameters satisfies the following restrictions

- \( b, \beta_i \in \mathbb{R}^m, \ i = 1, \ldots, m, \)
- \( \alpha_i \geq 0, \)
- \( m, M_i, \ i = 1, \ldots, m, \) Lévy measures.

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It is possible to define \( m + 1 \) independent Lévy processes \( Z(0), Z(1), \ldots, Z(m) \) taking values in \( \mathbb{R}^m \) with Lévy exponents \( F, R_1, \ldots, R_m \).

In [Kallsen, 2006] it has been proved that the time change equation

\[
X_t = x + Z(0)t + \sum_{k=1}^{m} Z(k) \int_0^t X(k) s \, ds,
\]

\( t \geq 0 \),

admits a weak solution, i.e. there exists a probability space containing two processes \((X, Z)\) such that (*) holds in distribution.

Moreover \( X \) has the distribution of a multi–type CBI with immigration mechanism \( F \) and branching mechanism \((R_1, \ldots, R_m)\).

Does there exist a pathwise solution of (*)?
Remarks

- It is possible to define $m + 1$ independent Lévy processes $Z^{(0)}, Z^{(1)}, \ldots, Z^{(m)}$ taking values in $\mathbb{R}^m$ with Lévy exponents $F, R_1, \ldots, R_m$. 

In [Kallsen, 2006] it has been proved that the time change equation

$$X_t = x + Z^{(0)} t + m \sum_{k=1}^{m} Z^{(k)} \int_0^t X^{(k)} s \, ds, \quad t \geq 0,$$

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In [Kallsen, 2006] it has been proved that the time change equation

$$X_t = x + Z_t^{(0)} + \sum_{k=1}^{m} Z^{(k)} \int_0^t X_s^{(k)} ds, \quad t \geq 0,$$  

admits a weak solution, i.e. there exists a probability space containing two processes $(X, Z)$ such that (*) holds in distribution.
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Results for multi–type CB

Theorem [G. and Teichmann, 2014]

Let $Z^{(1)}, \ldots, Z^{(m)}$ be independent $\mathbb{R}^m$-valued Lévy processes with

$$\mathbb{E}^0 \left[ e^{\langle u, Z^{(k)}_t \rangle} \right] = e^{tR_k(u)}, \quad u \in \mathcal{U},$$

where each $R_k$ is of LK form with triplets given by a set of admissible parameters. Then the time–change equation

$$X_t = x + \sum_{k=1}^{m} Z^{(k)}_t \int_0^t X^{(k)}_s \, ds \quad t \geq 0,$$

adopts a unique solution, which is a multi–type CB process with respect to the time–changed filtration.
Multiparameter time–change filtration

- Define
  \[ Z = (Z_1^{(1)}, \ldots, Z_m^{(1)}, \ldots, Z_1^{(m)}, \ldots, Z_m^{(m)}) =: (Z^{(1)}, \ldots, Z^{(m^2)}) \].

- For all \( \underline{s} = (s_1, \ldots, s_{m^2}) \in \mathbb{R}_{\geq 0}^{m^2} \)
  \[ G_{\underline{s}}^h := \sigma \left( \{Z_{th}^{(h)}, \ t_h \leq s_h, \ \text{for} \ h = 1, \ldots, m^2\} \right) \].

- Complete it by \( G_{\underline{s}} = \bigcap_{n \in \mathbb{N}} G_{\underline{s(n)} + \frac{1}{n}} \lor \sigma(\mathcal{N}) \).

Definition

A random variable \( \underline{\tau} = (\tau_1, \ldots, \tau_{m^2}) \in \mathbb{R}_{\geq 0}^{m^2} \) is a \((G_{\underline{s}})\)-stopping time if

\[ \{\tau \leq s\} := \{\tau_1 \leq s_1, \ldots, \tau_{m^2} \leq s_{m^2}\} \in G_{\underline{s}}, \ \text{for all} \ s \in \mathbb{R}_{\geq 0}^{m^2} \).

If \( \underline{\tau} \) is a stopping time,

\[ G_{\underline{\tau}} := \{B \in G \mid B \cap \{\tau \leq s\} \in G_{\underline{s}} \ \text{for all} \ s \in \mathbb{R}_{\geq 0}^{m^2}\} \].
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Extension to multi–type CBI

Let $F$ be an immigration mechanism and $R = (R_1, \ldots , R_m)$ a branching mechanism.

Let $Z^{(0)}$ Lévy process with exponent $F$ and $Z^{(i)}$ Lévy process with exponent $R_i$.

Define, for $k = 0, \ldots , m$,

$$Z^{(k)} := (\underbrace{Z_{0}^{(k)}, Z_{1}^{(k)}, \ldots , Z_{m}^{(k)}}_{m \text{ coordinates}}) := (0, \underbrace{Z^{(k)}}_{m \text{ coordinates}}).$$

Given $y = (1, x)$ with $x \in \mathbb{R}_{\geq 0}^m$, the previous result gives pathwise existence of

$$Y_t = y + \sum_{k=0}^{m} \int_{0}^{t} Z^{(k)} \, Y_{s}^{(i)} \, ds.$$

It holds $Y = (1, X)$ where $X$ is a CBI($F, R$).
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The one dimensional case

**Question:** Given a Lévy process $Z$ taking values in $\mathbb{R}$, is there a solution of

$$X_t = x + \int_0^t X_s \, ds$$

For the one dimensional case see also [Caballero et al., 2013].
An ODE point of view in $\mathbb{R}_{\geq 0}$

Introduce

$$\tau(t) := \int_0^t X_s ds.$$  

Does there exist a solution $\tau \in \mathbb{R}_{\geq 0}$ of

$$\begin{cases} 
\dot{\tau}(t) = x + Z(\tau(t)) \\
\tau(0) = 0 
\end{cases}$$
An ODE point of view in $\mathbb{R}^m_{\geq 0}$

Introduce

$$Z : \mathbb{R}^m_{\geq 0} \rightarrow \mathbb{R}^m$$

$$s \mapsto \sum_{i=1}^{m} Z^{(i)}(s_i).$$

Does there exist a solution $\tau \in \mathbb{R}^m_{\geq 0}$ of

$$\begin{cases}
\dot{\tau}(t) = x + Z(\tau(t)), \\
\tau(0) = 0.
\end{cases}?$$
Construction of the time–change process

Theorem [G. and Teichmann, 2014]

There exists a solution of

\[
\begin{align*}
\dot{\tau}((t_0, \tau_0, x); t) &= (x + \mathcal{Z})(\tau((t_0, \tau_0, x); t)), \\
\tau((t_0, \tau_0, x); t_0) &= \tau_0,
\end{align*}
\]

for \( t \geq t_0 \) and \( \tau_0 \in \mathbb{R}^m_{\geq 0} \).
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Theorem [G. and Teichmann, 2014]

There exists a solution of

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Construction of the time–change process

**Theorem [G. and Teichmann, 2014]**

There exists a solution of

\[
\begin{align*}
\dot{\tau}((t_0, \tau_0, x); t) &= (x + Z)(\tau((t_0, \tau_0, x); t)), \\
\tau((t_0, \tau_0, x); t_0) &= \tau_0,
\end{align*}
\]

for \( t \geq t_0 \) and \( \tau_0 \in \mathbb{R}_{\geq 0}^m \).
Decomposition of $Z$

The Lévy–Itô decomposition together with the **canonical form** of the admissible parameters give

$$Z_t^{(i)} = \beta_i t + \sigma_i B_t^{(i)} + \int_0^t \int \xi \mathbf{1}_{\{|\xi| > 1\}} \mathcal{J}^{(i)}(d\xi, ds)$$

$$+ \int_0^t \int \xi \mathbf{1}_{\{|\xi| \leq 1\}} \left( \mathcal{J}^{(i)}(d\xi, ds) - M_i(d\xi) ds \right)$$

where $\sigma_i = \sqrt{\alpha_i}$, $B^{(i)}$ is a process in $\mathbb{R}^m$ which evolves only along the the $i$-th coordinate as Brownian motion and $\mathcal{J}^{(i)}$ is the jump measure of the process $Z^{(i)}$. 
Decompose

\[ Z^{(i)} = \tilde{Z}^{(i)} + \hat{Z}^{(i)} \]

where \( \tilde{Z}^{(i)} \) and \( \hat{Z}^{(i)} \) are two stochastic processes on \( \mathbb{R}^m \) defined by

\[
\tilde{Z}^{(i)}(t) := (\beta_i)_i t + \sigma_i B^{(i)}(t) + \int_0^t \int \xi_i 1_{\{|\xi|>1\}} \mathcal{J}^{(i)}(d\xi, ds) \\
+ \int_0^t \int \xi_i 1_{\{|\xi|\leq 1\}} \tilde{\mathcal{J}}^{(i)}(d\xi, ds)
\]

\[
\hat{Z}^{(i)}(t) := 0, \quad \text{for } k \neq i,
\]

\[
\tilde{\mathcal{J}}^{(i)}(t) := \tilde{\beta}_i t + \int_0^t \int (\xi - \xi_i e_i) 1_{\{|\xi|>1\}} \mathcal{J}^{(i)}(d\xi, ds) \\
+ \int_0^t \int (\xi - \xi_i e_i) 1_{\{|\xi|\leq 1\}} \tilde{\mathcal{J}}^{(i)}(d\xi, ds).
\]

where \( \tilde{\beta}_i = \beta_i - e_i(\beta_i)_i \) and \( \tilde{\mathcal{J}}^{(i)} \) is the compensated jump measure.
Approximation of the jump part

- Introduce, for all \( s \in \mathbb{R}^m_{\geq 0} \),

\[
\tilde{Z}(s) := \sum_{i=1}^{m} \tilde{Z}^{(i)}(s_i), \quad \overline{Z}(s) := \sum_{i=1}^{m} \overline{Z}^{(i)}(s_i).
\]

- Fix \( M \in \mathbb{N} \) and consider the partition

\[
\mathcal{T}_M := \left\{ \frac{k}{2^M}, \quad k \geq 0 \right\}.
\]

- Define the following approximations on the partition \( \mathcal{T}_M \):

\[
\uparrow \tilde{Z}^{(i, M)}_t := \sum_{k=0}^{\infty} \tilde{Z}^{(i)}_{k/2^M} 1_{\left[ \frac{k}{2^M}, \frac{k+1}{2^M} \right)}(t),
\]

\[
\uparrow \tilde{Z}^{(M)}(s) := \sum_{i=1}^{m} \uparrow \tilde{Z}^{(i, M)}(s_i).
\]
The proof

■ Set

\[
(t_0, \tau_0, x) := (0, 0, x), \\
\dot{\overrightarrow{\sigma}} := (0, \ldots, 0), \\
\overrightarrow{\sigma} := (\sigma^{(1, M)}_1, \ldots, \sigma^{(i, M)}_1, \ldots, \sigma^{(m, M)}_1)
\]

■ Solve

\[
\begin{cases}
\dot{\tau}((0, 0, x); t) = \left(x + \tilde{Z}\right)(\tau((0, 0, x); t)), \\
\tau((0, 0, x); t_0) = \tau_0,
\end{cases}
\]

for \( t \in [0, t_1] \) where

\[
t_1 := \sup\{ t > 0 \mid \tau((t_0, \tau_0, x); t) \leq \overrightarrow{\sigma}\}.
\]

Remark  There might be one or more indices \( i^* \), where equality holds. Collect them in a set \( I^* \subseteq \{1, \ldots, m\} \).
Update the values

\[ \pi_{I^*} \leftarrow \sigma := \pi_{I^*} \rightarrow \sigma, \]
\[ \pi_{I^*} \rightarrow \sigma := \pi_{I^*} \rightarrow \sigma ++, \]

where \( \rightarrow \sigma ++ \) contains the next jumps of \( \uparrow \sim^{(i, M)} \) for all \( i \in I^* \) after \( \rightarrow \sigma_i \).

Define

\[ \tau_1 := \tau((t_0, \tau_0, x); t_1) \]
\[ x_1 := x + \Delta \uparrow \sim^{(M)}(\rightarrow \sigma). \]

Solve

\[
\begin{cases}
\dot{\tau}((t_1, \tau_1, x_1); t) = (x_1 + \tilde{\mathcal{Z}})(\tau((t_1, \tau_1, x_1); t)), \\
\tau((t_1, \tau_1, x_1); t_1) = \tau_1,
\end{cases}
\]

for \( t \in [t_1, t_2] \) where

\[ t_2 := \sup \{ t > t_1 \mid \tau((t_1, \tau_1, x_1); t) \leq \rightarrow \sigma \}. \]
Define iteratively, for all \( n \geq 1 \)

\[
\begin{align*}
  t_{n+1} &:= \sup \{ t > 0 \mid \tau((t_n, \tau_n, x_n); t) \leq \sigma \}, \\
  \tau_{n+1} &:= \tau((t_n, \tau_n, x_n); t_{n+1}), \\
  x_{n+1} &:= x_n + \Delta^\uparrow \hat{Z}(M)(\sigma),
\end{align*}
\]

where, at each step $\sigma$ and $\sigma$ are updated.
Solution of the approximated problem

Theorem

There exists a solution of

\[
\begin{cases} \dot{\tau}^{(M)}((t_0, \tau_0, x); t) = (x + \sim Z + ^\uparrow \sim Z^{(M)})(\tau^{(M)}((t_0, \tau_0, x); t)), \\ \tau^{(M)}((t_0, \tau_0, x); t_0) = 0. \end{cases}
\]

Moreover it holds

\[
\lim_{M \to \infty} \tau^{(M)}((t_0, \tau_0, x); t) = \tau((t_0, \tau_0, x); t)
\]

where \(\tau\) solves

\[
\begin{cases} \dot{\tau}((t_0, \tau_0, x); t) = (x + Z)(\tau((t_0, \tau_0, x); t)), \\ \tau((t_0, \tau_0, x); t_0) = \tau_0. \end{cases}
\]
Thank you for your attention
Stochastic differential equation with jumps for multi-type continuous state and continuous time branching processes with immigration.

A Lamperti-type representation of continuous-state branching processes with immigration.

Affine processes and applications in finance.

Pathwise construction of affine processes.

A didactic note on affine stochastic volatility models.