Continuum Widom-Rowlinson model

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1. Poisson point process

2. Widom-Rowlinson model on a bounded region $\Lambda$

3. Infinite volume Widom-Rowlinson model
Definition 1

Let $S$ be a metric space embedded with the borel $\sigma$-algebra $S$ (Think $S=\mathbb{R}^2$).
Let $M := M_S$ be the space of locally finite subsets of $S$. $M$ is equipped with the smallest $\sigma$-algebra $\mathcal{M}$ for which the counting variables

$$\gamma \in M \mapsto \#(\gamma \cap \Delta)$$

are measurable for all bounded subset $\Delta$ of $S$. 
Definition 2

Let $m$ be a positive measure in $S$. A **Poisson point process** of intensity measure $m$ is a measurable function $X$ from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(M, \mathcal{M})$ which satisfies: for all integer $n$, for all $\Delta_1, \ldots, \Delta_n$ disjoint bounded subsets of $S$ and for all integers $k_1, \ldots, k_n$, we have

$$
\mathbb{P}(\#(X \cap \Delta_1) = k_1, \ldots, \#(X \cap \Delta_n) = k_n) = \prod_{i \in \{1, \ldots, n\}} \frac{e^{-m(\Delta_i)} m(\Delta_i)^{k_i}}{k_i!}.
$$

Simulation of a Poisson Point Process in a region $\Delta$:

- Choose the number of points according to a Poisson distribution of mean $m(\Delta)$.
- Given the number of points, each point is independently distributed by the probability law $\frac{m(\cdot \cap \Delta)}{m(\Delta)}$. 
Notations

- $S = \mathbb{R}^2 \times \mathbb{R}^+ \times \{1, 2\}$. A element of this space will be seen has a colored ball.
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- \( m = z \times \text{lebesgue} \otimes Q \otimes \mathcal{U}(\{1, 2\}) \), where \( z > 0 \), \( Q \) is a probability law on \( \mathbb{R}^+ \) and \( \mathcal{U}(\{1, 2\}) = \frac{1}{2}(\delta_1 + \delta_2) \).

**Simulation of poisson point process in the bounded box \( \Lambda \subseteq \mathbb{R}^2 \):**

1. Choose the number of points \( N \sim \mathcal{P}(z|\Lambda|) \).
2. Given the number of points, each are uniformly and independently distributed in \( \Lambda \).
3. In each point, choose a random radius independently and according to \( Q \).
4. In each point, choose a color independently and uniformly over the 2 colors.
For a subset $\Lambda$ of $\mathbb{R}^2$ and a element $\gamma$ of $M$, $\gamma_{\Lambda}$ denote the element of $M$ which contain all the balls of $\gamma$ centered inside $\Lambda$. 

\begin{align*}
\gamma \\
\gamma_{\Lambda}
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We will denote by $\pi^z,Q$ the probability law on $(M,M)$ of a Poisson point process of intensity measure $m$.

$\pi^z,Q_\Lambda$ will denote the projection of $\pi^z,Q$ in $\Lambda \times \mathbb{R}^+ \times \{1, 2\}$, i.e.

$$\int_M f(\gamma) d\pi^z,Q_\Lambda(\gamma) = \int_M f(\gamma_\Lambda) d\pi^z,Q(\gamma).$$

(We only look at the balls centred inside $\Lambda$)
We want to construct a probability measure on $M$ for which balls of different color can’t intersect. So we define

$$\mathcal{A} = \{ \gamma \in M, \forall (x, R, k), (x', R', k') \in \gamma, k \neq k' \Rightarrow ||x-x'|| > R+R' \}$$
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**Proposition 3**

- $\pi^{z,Q}(\mathcal{A}) = 0$, so we can’t conditioned $\pi^{z,Q}$ by the event $\mathcal{A}$.
- For all bounded set $\Lambda$ of $\mathbb{R}^2$, $\pi^{z,Q}(\Lambda) > 0$. 
Definition 4

The widom-Rowlinson measure of parameter $z, Q$ on the bounded region $\Lambda$ is

$$P_{\Lambda}^{z, Q}(d\gamma) = \frac{I_A(\gamma)}{Z_\Lambda} \pi_{\Lambda}^{z, Q},$$

where $I_A$ is the indicator function of the set $A$ and $Z_\Lambda = \pi_{\Lambda}^{z, Q}(A)$ is the normalizing constant.
Definition 5

A probability measure $\mu$ on $M$ is called an (infinite volume) Widom-Rowlinson measure of parameters $z$ and $Q$ if it satisfies $\mu(\Lambda) = 1$ and the following DLR (Dobrushin, Landford, Ruelle) equations: For all bounded region $\Lambda$ of $\mathbb{R}^2$ and for all bounded function $f$

\[
\int_M f \, d\mu = \int_M \int_M f(\gamma' + \gamma^c) \frac{I_\Lambda(\gamma'_\Lambda + \gamma^c)}{Z(\gamma^c)} \pi^z,Q(d\gamma') \mu(d\gamma).
\]
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Theorem 6 (H-Dereudre)

- If $\int r^2 Q(dr) < \infty$, then for all positive $z$ there exist a Widom-Rowlinson measure of parameter $z, Q$.
- In the case $\int r^2 Q(dr) = \infty$, there exist a critical value $z_c > 0$ such that, for all $0 < z < z_c$, there exist a Widom-Rowlinson measure of parameters $z, Q$. 
Step 1 : find a good candidate

- Let $\Lambda_n$ be the region $]-n, n]^2$
- $\mu_n(d\gamma) := \mu_n^{z,Q}(d\gamma) = \frac{I_{\Lambda}(\gamma)}{Z_n} \pi_{\Lambda_n}^{z,Q}(d\gamma) = P_{\Lambda_n}^{z,Q}(d\gamma)$
Step 1: find a good candidate

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- $\mu_n(d\gamma) := \frac{I_{\Lambda_n}(\gamma)}{Z_n} \pi_{\Lambda_n}^{Z,Q}(d\gamma) = P_{\Lambda_n}^{Z,Q}(d\gamma)$.
- Let $\tilde{\mu}_n$ be the probability measure on $M$ such that the configuration on each box $\Lambda_n + x$, $x \in 2n\mathbb{Z}^2$ are independent and identically distribute according to $\mu_n$.
- Finally $\bar{\mu}_n(d\gamma) = \frac{1}{\#(\mathbb{Z}^2 \cap \Lambda_n)} \times \sum_{i \in \mathbb{Z}^2 \cap \Lambda_n} \tilde{\mu}_n \circ v_i^{-1}(d\gamma)$, where $v_i$ is the translation of the vector $i$.

Using the good tools, we will show that the sequence $(\bar{\mu}_n)$ have a cluster point, for a good topology.
Definition 7

Let $\mathcal{P}$ denote the set of probability measures on $M$. We say that a sequence $(\nu_n)$ in $\mathcal{P}$ converge to $\nu$ for the local convergence topology if, for all local bounded function $f$, we have

$$\int_M f \, d\nu_n \to \int_M f \, d\nu,$$

where a local function is a function that satisfies, for a bounded $\Lambda$, $f(\gamma) = f(\gamma\Lambda)$. 

Definition 8

Let $\mathcal{P}_\theta$ denote the set of probability measure on $M$ which are invariant for the translations $(\nu_x)_{x \in \mathbb{Z}^2}$.

For $\nu$ in $\mathcal{P}_\theta$, we define the specific entropy of $\nu$,

$$I(\nu) = \lim_{n \to \infty} \frac{1}{|\Lambda_n|} I_{\Lambda_n}(\nu|\pi^z,Q),$$

where

$$I_{\Lambda_n}(\nu|\pi^z,Q) = \begin{cases} \int_M f \ln(f) d\pi^z_Q & \text{if } \nu_{\Lambda_n} \ll \pi^z_{\Lambda_n}, \\ +\infty & \text{sinon} \end{cases}$$

Proposition 9 (Georgii 1988)

The specific entropy is affine, i.e. 

$$I(\alpha \nu + (1-\alpha) \tilde{\nu}) = \alpha I(\nu) + (1-\alpha) I(\tilde{\nu})$$

For all $C > 0$, the set 

$$\{ \nu \in \mathcal{P}_\theta, I(\nu) \leq C \}$$

is compact for the local convergence topology.
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- For all $C > 0$, the set $\{\nu \in \mathcal{P}_\theta, I(\nu) \leq C\}$ is compact for the local convergence topology.
Lemma 10

For all $n$, $I(\bar{\mu}_n) \leq z$, So the sequence admits a cluster point $\bar{\mu}$.

Lemma 11

- $\bar{\mu} \in \mathcal{P}_\theta$.
- $\bar{\mu}(A) = 1$. 
We want to show that, for a given bounded $\Lambda$ and for a given bounded function, we have:

$$\int_{M} f \, d\bar{\mu} = \int_{M} \int_{M} f(\gamma'_{\Lambda} + \gamma_{\Lambda}^{c}) \frac{I_{\Lambda}(\gamma'_{\Lambda} + \gamma_{\Lambda}^{c})}{Z(\gamma_{\Lambda}^{c})} \pi_{\Lambda}^{z,Q}(d\gamma') \bar{\mu}(d\gamma).$$
We want to show that, for a given bounded $\Lambda$ and for a given bounded function, we have:

$$\int_M f \, d\bar{\mu} = \int_M \int_M f(\gamma^{\Lambda} + \gamma^{\Lambda_c}) \frac{I_\Lambda(\gamma^{\Lambda} + \gamma^{\Lambda_c})}{Z(\gamma^{\Lambda_c})} \pi \, Q'(\gamma^{\Lambda}) \bar{\mu}(d\gamma).$$

Because of the $\sigma$-field structure, we can restrict ourself to the case of $f$ being local.
Case 1: Bounded radii

In the case of radii being bounded by a constant $C > 0$, and because we have $\bar{\mu}(A) = 1$, we have

$$I_A(\gamma'_A + \gamma_{A}^c) = I_A(\gamma'_A + \gamma_{A}^c \oplus B(0, C) \setminus A)$$

for $\bar{\mu}$ – almost every $\gamma$.

So we can directly use the local convergence of $\bar{\mu}_n$ to $\bar{\mu}$.
Case 2: $\int r^2 Q(dr) < \infty$

- Using stochastic domination methods, we can show that $\bar{\mu}(\mathcal{W}) = 1$ where $\mathcal{W} = \bigcup_{k \in \mathbb{N}} \mathcal{W}_k$ and

  $\mathcal{W}_k = \{ \gamma \in \mathcal{M}, \forall (x, R, k) \in \gamma, R \leq \frac{|x|}{2} + k \}.$

- So we can use the same idea as before to show the DLR equation.
Case 3: $\int r^2 Q(dr) = \infty$

In this case the stochastic domination methods don't apply any more.

Idea: To "make local" the term $l_A(\gamma'_\Lambda + \gamma_{\Lambda^c})$, we want to protect the balls in $\gamma'_\Lambda$ by putting a "shield" which will prevent the balls far to intersect the one in $\Lambda$.
For this shield to work, we need to have

$$\bar{\mu} \left( \{ \gamma \in M, \gamma \text{ got at least a ball of each color} \} \right) = 1.$$ 

Problem: I don’t know if that’s true!
Idea: Show that this probability is positive and conditioned on this event.
Comparing the specific entropy of $\bar{\mu}$ to the entropy of probabilities giving "monochromatic" configurations, we get

**Proposition 12**

*There is a $z_c > 0$ such that, for all $z < z_c$, we have*

$$\bar{\mu}\left(\{\gamma \in M, \gamma \text{ got at least a ball of each color}\}\right) > 0.$$  

So by conditioning on this event, we get a probability for which every configuration has at least one ball of each color.
Poisson point process
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