Simple Smiles
For The Mixing Setup
Joint work with D. Sloth

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Analytical approximations of implied volatility have been and continue to be proposed, even for solvable models, for the need of

- Transparency
- Robustness
- Speed

In this work, we propose an approximation of the implied volatility which can be used for a large variety of well-established models and is

- Transparent, as it decomposes the /smile into meaningful quantities associated with higher-order option risks.
- Simple, fast and easy to implement.
- Quite accurate where it matters.
The Mixing Setup
The Mixing Setup

- The mixing setup is a natural generalization of the Black-Scholes model

\[ S_t = S_0 \exp \left( -\frac{1}{2} \sigma^2 t + \mathcal{W}_{\sigma^2 t} \right) \]

obtained by randomizing the spot \( S_0 \) and the total variance \( \sigma^2 t \) via their stochastic counterparts.

- The risk-neutral dynamics of the asset price are given by

\[ S_t = S_{eff}^t \exp \left( -\frac{1}{2} V_{eff}^t + \mathcal{W}_{V_{eff}^t} \right), \quad V_{eff}^0 = 0, \quad S_{eff}^0 = S_0, \]

where \( S_{eff} \) is a positive martingale, \( V_{eff} \) is an increasing process and \( \mathcal{W} \) is a Brownian motion, independent of \( (S_{eff}, V_{eff}) \).

- The price process \( S \) is a conditionally log-normal martingale.
The mixing setup contains stochastic volatility models of the following type

\[
\begin{align*}
    dS_t &= S_t \sqrt{v_t} \left( \sqrt{1 - \rho^2} dW + \rho dB \right) \\
    dv_t &= \mu(v_t, t) dt + \sigma(v_t, t) dB_t, \\
    v_0 &> 0,
\end{align*}
\]

where \( W \perp B \) are Brownian motions and \( \rho \) is the correlation parameter.

The Heston model, the 3/2 model, and the quadratic class specification are examples of solvable specifications.

The mixing representation follows by setting

\[
\begin{align*}
    V_{\text{eff}}^t &= (1 - \rho^2) \int_0^t v_s ds, \\
    S_{\text{eff}}^t &= S_0 \exp \left( -\frac{\rho^2}{2} \int_0^t v_s ds + \rho \int_0^t \sqrt{v_s} dB_s \right),
\end{align*}
\]
Purely jumping models

- Purely jumping models are obtained by setting $V^{\text{eff}}$ as an increasing and purely jumping semimartingale.

- The price dynamics are then specified as

$$S_t = S_t^{\text{eff}} \exp \left( -\frac{1}{2} V_t^{\text{eff}} + W_{V_t^{\text{eff}}} \right),$$

$$S_t^{\text{eff}} = S_0 \exp \left( -K_t(c) + cV_t^{\text{eff}} \right),$$

where the real parameter $c$ allows for correlation between $S$ and $V^{\text{eff}}$, and $K(c)$ is the cumulant exponent process.

- **Exponential Lévy models** are obtained by modeling $V^{\text{eff}}$ as a drift-less Lévy subordinator.

- Models of this class include, for instance, the VG model, the NIG model, and the CGMY model.
The Two Series Expansions
The $\langle S, V \rangle$-expansion

By conditional log-normality, the price $C(S_0, K, \tau)$ of a call with strike $K$ and expiry $\tau > 0$ admits the mixing representation

$$C(S_0, K, \tau) = \mathbb{E}[(S_\tau - K)^+] = \mathbb{E}[C_{BS}(S_{\tau}^{\text{eff}}, V_{\tau}^{\text{eff}})],$$

where $C_{BS}(S, V)$ denotes the BS call-price in terms of total variance $V = \sigma \tau$.

First, we apply a 2-dimensional Taylor series expansion around the point $(\mathbb{E}S_{\tau}^{\text{eff}}, \mathbb{E}V_{\tau}^{\text{eff}}) = (S_0, \mathbb{E}V_{\tau}^{\text{eff}})$

$$\mathbb{E}[C_{BS}(S_{\tau}^{\text{eff}}, V_{\tau}^{\text{eff}})] = C_{BS}(S_0, \mathbb{E}V_{\tau}^{\text{eff}}) + \frac{1}{2!} \mathbb{E}[(S_{\tau}^{\text{eff}} - S_0)^2] \frac{\partial^2 C_{BS}}{\partial S^2}$$

$$+ \frac{1}{2!} \mathbb{E}[(V_{\tau}^{\text{eff}} - \mathbb{E}V_{\tau}^{\text{eff}})^2] \frac{\partial^2 C_{BS}}{\partial V^2}$$

$$+ \mathbb{E}[(S_{\tau}^{\text{eff}} - S_0)(V_{\tau}^{\text{eff}} - \mathbb{E}V_{\tau}^{\text{eff}})] \frac{\partial^2 C_{BS}}{\partial S \partial V} + \cdots$$
The $\langle \Sigma \rangle$-expansion

Next, recall that by definition

$$C(S_0, K, \tau) \equiv C_{BS}(S_0, \Sigma).$$

where $\Sigma = \tau I^2$ denotes the implied total variance.

Then expand this expression in the second variable $\Sigma$ around $E V_{\tau}^{\text{eff}}$ using a one-dimensional Taylor series.

$$C(S_0, K, \tau) = C_{BS}(S_0, E V_{\tau}^{\text{eff}}) + (\Sigma - E V_{\tau}^{\text{eff}}) \frac{\partial C_{BS}}{\partial V}$$

$$+ \frac{1}{2!} (\Sigma - E V_{\tau}^{\text{eff}})^2 \frac{\partial^2 C_{BS}}{\partial V^2} + \ldots$$

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Finally truncate the $\langle \Sigma \rangle$-expansion to the first order and the $\langle S, V \rangle$-expansion to the $q$-th order, to approximate $I^2$ as

$$ I^2 \approx \frac{\mathbb{E} V^\text{eff}_\tau}{\tau} + \frac{1}{\tau} \sum_{k=2}^{q} \sum_{l=0}^{k} \frac{\mathcal{D}_{s^l v^{k-l}}}{l!(k-l)!} \mathbb{E}[(S^\text{eff}_\tau - S_0)^l (V^\text{eff}_\tau - \mathbb{E} V^\text{eff}_\tau)^{k-l}], $$

where $\mathcal{D}_{s^m v^n} \equiv \left( \frac{\partial C_{BS}}{\partial V} \right)^{-1} \frac{\partial^{m+n} C_{BS}}{\partial S^m \partial V^n}$ denote the Vega-normalized Black-Scholes derivatives.

- Application demands that $\mathbb{E} \left[ (V_t^\text{eff})^m (S_t^\text{eff})^n \right]$ are easy to compute.

- This is a simple task whenever $\mathcal{L}_t^\text{eff}(u, w) = \mathbb{E} \left[ e^{u X_t^\text{eff} + w V_t^\text{eff}} \right]$ with $X^\text{eff} = \log S^\text{eff}$, is available in (semi) closed-form. In this case

$$ \mathbb{E} \left[ (V_t^\text{eff})^m (S_t^\text{eff})^n \right] = \frac{\partial^m \mathcal{L}_t^\text{eff}(u, w)}{\partial w^m} \bigg|_{u=n, w=0}, $$
A Simple Quadratic Approximation

- The 2nd-order expansion yields a simple approximation $I_Q^2(x, \tau)$ of the implied variance which is quadratic in $x = \log K/S$.

Specifically

$$I^2 \approx I_Q^2(x, \tau) = \frac{E V_{T}^{\text{eff}}}{\tau} + \frac{\text{Var}[S_{T}^{\text{eff}}]}{2\tau} D_{ss} + \frac{\text{Cov}[S_{T}^{\text{eff}}, V_{T}^{\text{eff}}]}{\tau} D_{sv} + \frac{\text{Var}[V_{T}^{\text{eff}}]}{2\tau} D_{vv},$$

where the normalized Gamma $D_{ss}$, Vanna $D_{sv}$ and Volga $D_{vv}$ are

$$D_{ss} = \frac{2}{S^2}, \quad D_{sv} = \frac{x}{SV} + \frac{1}{2S}, \quad D_{vv} = \frac{x^2}{2V^2} - \frac{1}{8} - \frac{1}{2V}.$$

- We see that
  - the Gamma risk $D_{ss}$ only contributes to the level of smile,
  - the Vanna term $D_{sv}$ determines the slope
  - the Volga term $D_{vv}$ introduces convexity.
A first look at ATM term-structure

Re-arranging the terms we obtain

\[ I_Q^2(x, \tau) = I_0(\tau) + I_1(\tau) x + I_2(\tau) x^2, \]

where \( I_\cdot(\tau) \) describe the term structure of the approximate smile:

- **ATM Variance**:
  \[
  I_0(\tau) = \frac{\mathbb{E} V_{\tau}^{\text{eff}}}{\tau} + \frac{\text{Var}[S_{\tau}^{\text{eff}}]}{S_0^2 \tau} + \frac{\text{Cov}[S_{\tau}^{\text{eff}}, V_{\tau}^{\text{eff}}]}{2 S_0 \tau} - \frac{\text{Var}[V_{\tau}^{\text{eff}}]}{4 \tau \mathbb{E} V_{\tau}^{\text{eff}}} \left( 1 + \frac{1}{4} \mathbb{E} V_{\tau}^{\text{eff}} \right)
  \]

- **ATM Skew**:
  \[
  I_1(\tau) = \frac{1}{\tau} \frac{\text{Cov}[S_{\tau}^{\text{eff}}, V_{\tau}^{\text{eff}}]}{S_0 \mathbb{E} V_{\tau}^{\text{eff}}}
  \]

- **ATM Curvature**:
  \[
  I_2(\tau) = \frac{1}{\tau} \frac{\text{Var}[V_{\tau}^{\text{eff}}]}{4 (\mathbb{E} V_{\tau}^{\text{eff}})^2}
  \]
Does it work?
Illustration for SV Models

Naturally, we consider the Heston (1993) model

\[ dv_t = \kappa(\theta - v_t)dt + \varepsilon v_t^{1/2} dB_t. \]

We also consider the 3/2 model with the instantaneous variance

\[ dv_t = v_t\kappa(\theta - v_t)dt + \varepsilon v_t^{3/2} dB_t. \]

- Both models are solvable, as the joint Laplace transform \( \mathcal{L}_t^{XV} \) of \( X = \log S \) and \( V = \int_0^t v_s ds \) has closed-form.
- Also the relevant moments are computable, since it holds that

\[ \mathcal{L}_t^{\text{eff}}(u, w) = \mathcal{L}_t^{XV} \left( u, (1 - \rho^2)(w + \frac{1}{2} u - \frac{1}{2} u^2) \right), \]

between the "standard" and the "effective" transforms.
- However, Fourier inversion is numerically not trivial in the 3/2 model, due to complex evaluations of the confluent hypergeometric function.
ATM vols (left) and skews (right) for the Heston (top) and the 3/2 (bottom). The maturity ranges from $\tau = 0.05$ up to $\tau = 18$ years. Parameters are as in Forde et al. (2012).
Does it work?

SV Models: The Smile at Short Maturities

The ATM accuracy of $I^2_Q(x, \tau)$ at short maturities is not coincidental.

- For models with time-independent coefficients

$$dv_t = a(v_t)dt + b(v_t)dB_t.$$ 

it holds that

$$\lim_{\tau \to 0} I^2(0, \tau) = v_0 \quad \text{and} \quad \lim_{\tau \to 0} \frac{\partial I^2}{\partial x^2} \bigg|_{x=0} = \frac{1}{2} \frac{\rho b(v_0)}{\sqrt{v_0}}$$

see e.g., Lewis (2000), Lee (2001) and Medvedev and Scaillet (2007).

- The quadratic approximation is consistent with these results

$$\lim_{\tau \to 0} \mathcal{I}_0(\tau) = v_0 \quad \text{and} \quad \lim_{\tau \to 0} \mathcal{I}_1(\tau) = \frac{1}{2} \frac{\rho b(v_0)}{\sqrt{v_0}}$$

- It is therefore tempting to compare $I^2_Q(x, \tau)$ with well-established asymptotic results.
Heston model at short maturities

- $I^2_Q$ vs Forde et al. (2012), for Heston model at short maturities.
- Maturities: $\tau = 0.05$ to $\tau = 0.5$. Moneyness: $\pm 15\%$. 
Does it work?

3/2 Model at Short Maturities

$I^2_Q$ vs Medvedev and Scaillet (2007), for general SV (plus jumps), short-maturities/small strikes. Maturities: $\tau = 0.07$ to $\tau = 0.5$. Moneyness: $\pm 15\%$. 
Mid-Long Maturities

- Top: Heston model. Bottom: 3/2 model
- Maturities $\tau = 1, 3, 5$. Moneyness: from $\pm 30\%$ to $\pm 40\%$. 

Does it work?
SV models: The smile at long maturities

At large maturities, the asymptotic behavior is

\[ \lim_{\tau \to \infty} I_2(x, \tau) = 8\lambda(k_0) \quad \text{and} \quad \frac{\partial I_2}{\partial x} \bigg|_{x=0} \approx -\frac{8ik_0 + 4}{\tau} + O(1/\tau^2) \quad \text{as} \quad \tau \to \infty \]

where \( \lambda \) is the first eigenvalue of a differential operator and \( k_0 \) is a complex number. See Lewis (2000), Jaquier (2007) and Tehranchi (2009).

For \( I_Q^2 \), if \( \rho \neq 0 \), it holds that

\[ \lim_{\tau \to \infty} I_Q^0(\tau) = \infty \quad \lim_{\tau \to \infty} I_Q^1(\tau) = 0 \quad \lim_{\tau \to \infty} I_Q^2(\tau) = 0 \]

So the accuracy of the quadratic approximation is bound to deteriorate as the maturity increases.

Luckily, this happens at a very slow rate, and the mismatch becomes observable only at extremely long expiries.
ATM Vols (left) and Skews (right), for Heston (top) and 3/2 (bottom), but with added ATM asymptotic behaviors. Maturity from $\tau = 0.05$ to $\tau = 18$ years.
Exponential Lévy models
Exponential Lévy Models

- **True short-maturity behavior**
  \[
  \lim_{\tau \to 0} \mathcal{I}^2(0, \tau) = 0 \quad \text{and} \quad \lim_{\tau \to 0} \mathcal{I}^2(x, \tau) = +\infty, \quad \text{for } x \neq 0
  \]

- **True long-maturity behavior**
  \[
  \lim_{\tau \to \infty} \mathcal{I}^2(x, \tau) = A \quad \text{and} \quad \mathcal{I}^2(x, \tau) \approx A + \frac{B}{\tau} + \frac{C}{\tau} x \quad \text{for large } \tau
  \]

- **In case of exponential Lévy models,** \( \mathcal{I}^2_Q(x, \tau) \) takes the form
  \[
  \mathcal{I}^2_Q(x, \tau) = \mathcal{A}(\tau) + \frac{B}{\tau} + \frac{C}{\tau} x + \frac{D}{\tau^2} x^2,
  \]
  with \( B < 0, \lim_{\tau \to 0} \mathcal{A}(\tau) = \tilde{A} \) and \( \lim_{\tau \to \infty} \mathcal{A}(\tau) = \infty \) (unless \( c = 0 \)).

  - In spite of this ”anti-asymptotic” behavior, \( \mathcal{I}^2_Q(x, \tau) \) is nevertheless quite useful.
VG Model smiles

\( I_2^Q(x, \tau) \) vs Jäckel (2009) singular approximation. Maturity: \( \tau = 0.25 \) up to \( \tau = 5 \). Moneyness: \( \pm 30\% \) up to \( \pm 50\% \). Jäckel (2009) parameters.
While both approximations are singular around the expiry-date, they both capture the overall behavior of the surface for a large relevant region of the smile.
Concluding Remarks

- The approximation is simple, easy to implement, and quite accurate where it matters (i.e. liquid moneyness).
- The approximation decomposes the volatility smile into meaningful quantities associated with higher-order option risks.
- The approximation is largely generic in the sense that it may be used for a large variety of option pricing models.
- Finally, in the paper we explore two domains of application of the approximation.
  1. We suggest to use the approximation as a control variate in Fourier option pricing.
  2. We propose a ‘speedy’, approximation-based approach for model calibration to at-the-money volatilities and skews.