A population dynamic approach to rough Heston models

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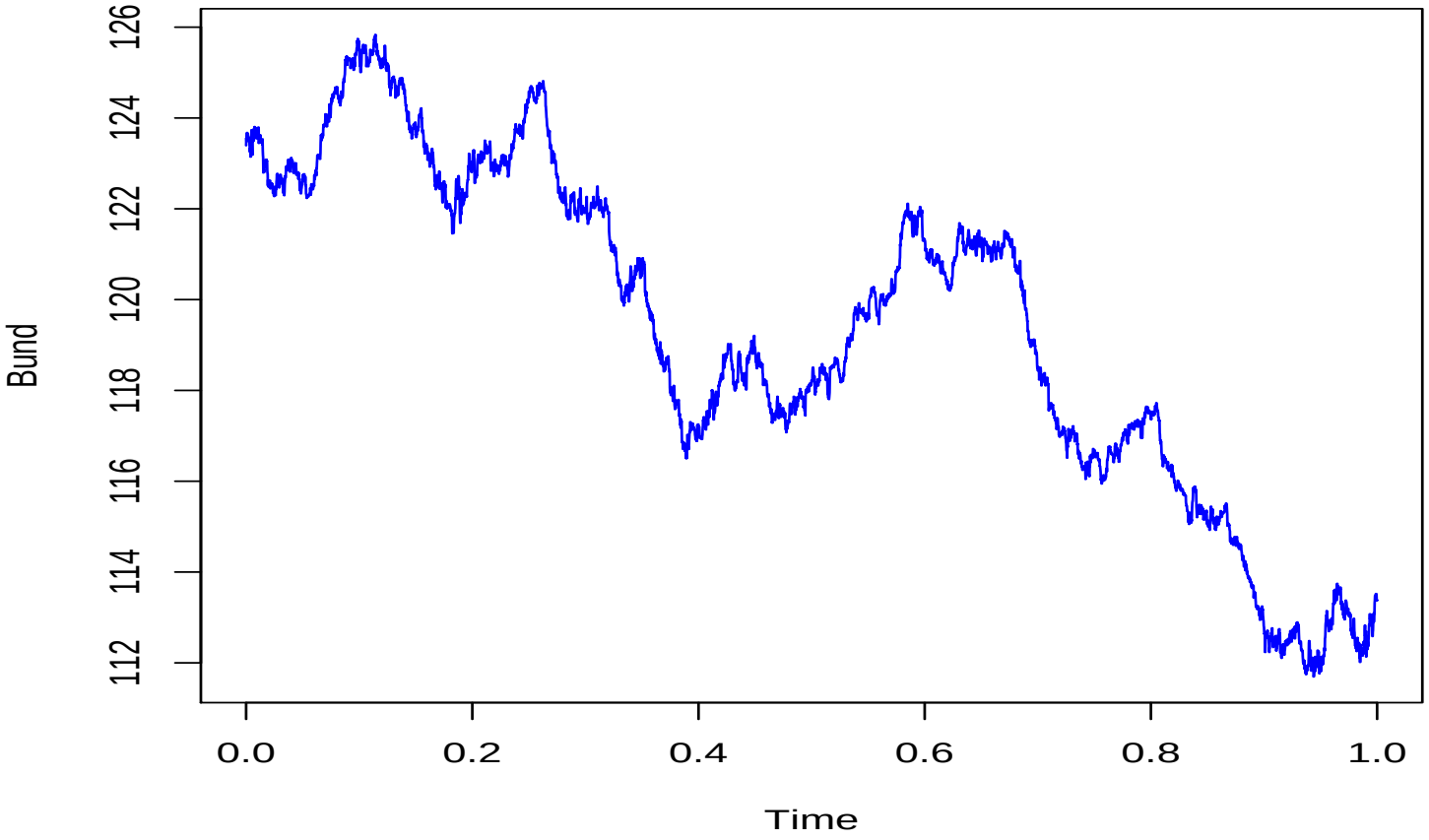
One financial asset, 2 years



A well-know stochastic volatility model

The Heston model

A very popular stochastic volatility model for a stock price is the Heston model :

$$dS_t = S_t \sqrt{V_t} dW_t$$

$$dV_t = \lambda(\theta - V_t)dt + \lambda\nu\sqrt{V_t}dB_t, \quad \langle dW_t, dB_t \rangle = \rho dt.$$

Popularity of the Heston model

- Reproduces several important features of low frequency price data : leverage effect, time-varying volatility, fat tails,. . .
- Provides quite reasonable dynamics for the volatility surface.
- Explicit formula for the characteristic function of the asset log-price$\rightarrow$ very efficient model calibration procedures.

But...Volatility is rough !

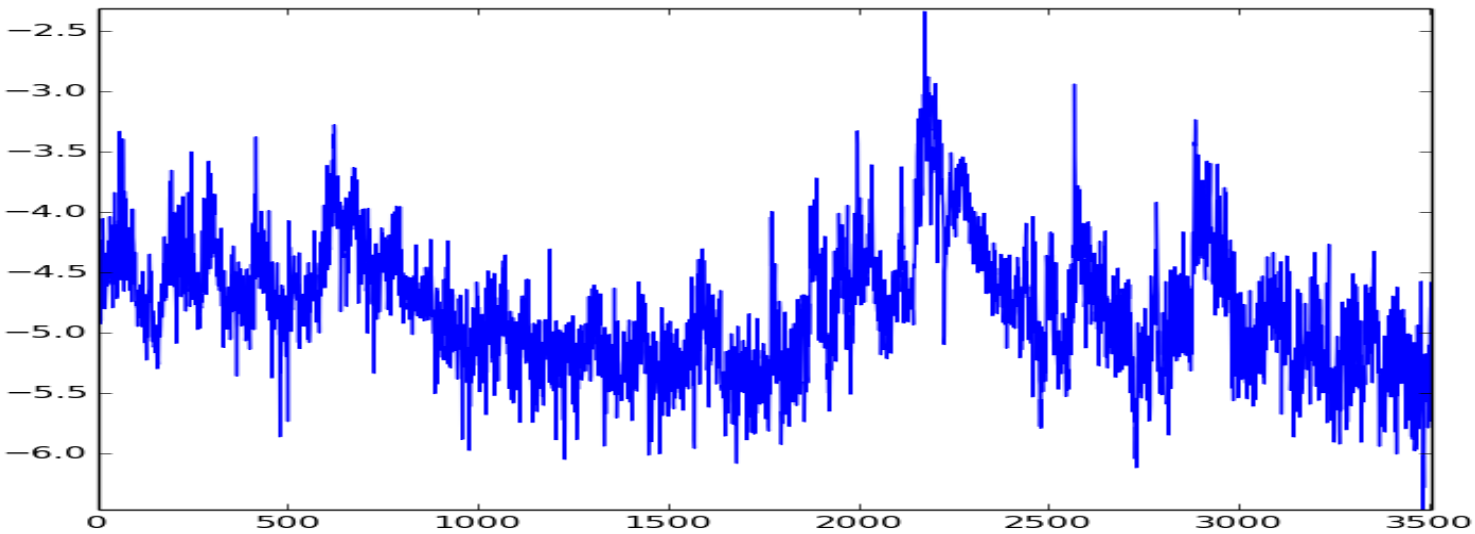


FIGURE : The log volatility $\log(\sigma_t)$ as a function of t, S&P, 3500 days.

Rough volatility

Volatility as a rough fractional process

- In Heston model, volatility follows a Brownian diffusion.
- It is shown in Gatheral *et al.* that log-volatility time series behave in fact like a fractional Brownian motion, with Hurst parameter of order 0.1.
- More precisely, basically all the statistical stylized facts of volatility are retrieved when modeling it by a rough fractional Brownian motion.
- From Alos, Fukasawa and Bayer *et al.*, we know that such model also enables us to reproduce very well the behavior of the implied volatility surface, in particular the at-the-money skew (without jumps).

Fractional Brownian motion (I)

Definition

The fractional Brownian motion with Hurst parameter H is the only process W^H to satisfy :

- Self-similarity : $(W^H_{at}) \overset{\mathcal{L}}{=} a^H (W^H_t)$.
- Stationary increments : $(W^H_{t+h} - W^H_t) \overset{\mathcal{L}}{=} (W^H_h)$.
- Gaussian process with $\mathbb{E}[W^H_1] = 0$ and $\mathbb{E}[(W^H_1)^2] = 1$.

Fractional Brownian motion (II)

Proposition

For all $\varepsilon > 0$, W^H is $(H-\varepsilon)$-Hölder a.s.

Proposition

The absolute moments satisfy

$$\mathbb{E}[|W^H_{t+h} - W^H_t|^q] = K_q h^{Hq}.$$

Mandelbrot-van Ness representation

$$W^H_t = \int_0^t \frac{dW_s}{(t-s)^{\frac{1}{2}-H}} + \int_{-\infty}^0 \left(\frac{1}{(t-s)^{\frac{1}{2}-H}} - \frac{1}{(-s)^{\frac{1}{2}-H}} \right) dW_s.$$

Evidence of rough volatility

Volatility of the S&P

- Everyday, we estimate the volatility of the S&P at 11am (say), over 3500 days.
- We study the quantity

$$m(\Delta, q) = \mathbb{E}[|\log(\sigma_{t+\Delta}) - \log(\sigma_t)|^q],$$

  for various q and Δ, the smallest Δ being one day.
- In the case where the log-volatility is a fractional Brownian motion : $m(\Delta, q) \sim c\Delta^{qH}$.

Example : Scaling of the moments

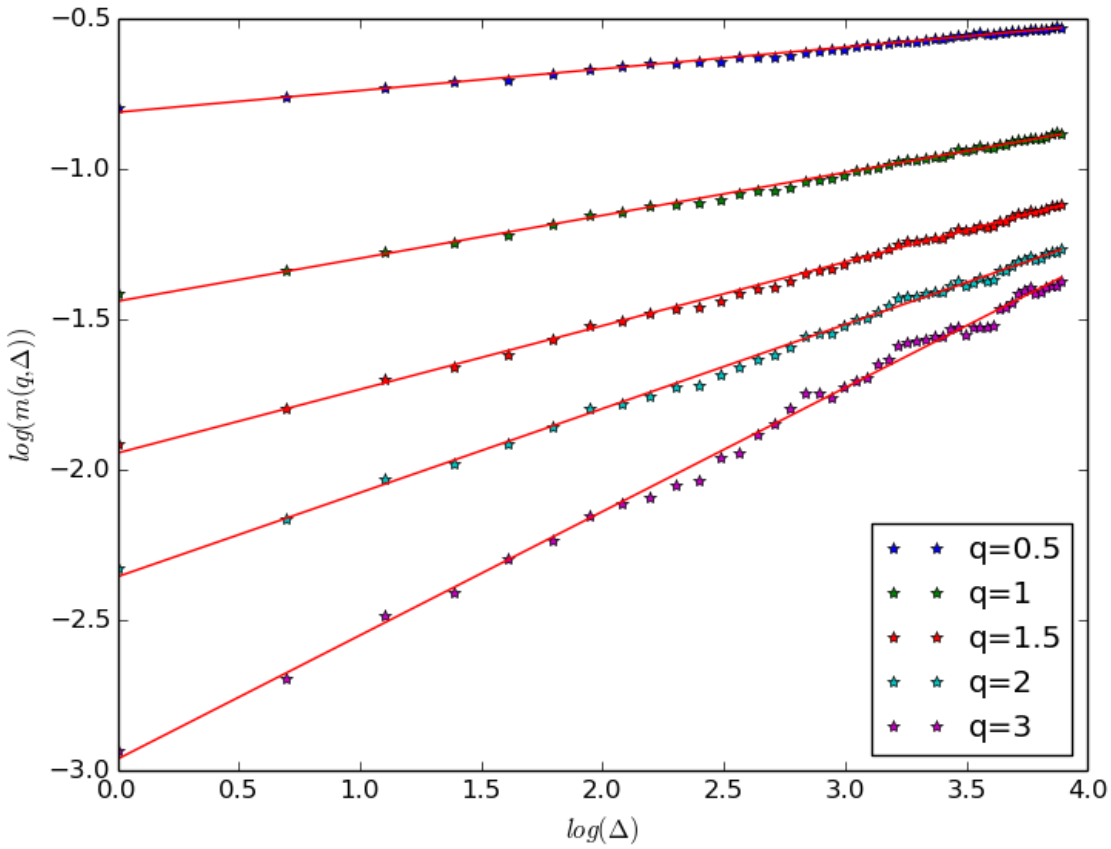


FIGURE : $\log(m(q,\Delta)) = \zeta_q \log(\Delta) + C_q$. The scaling is not only valid as Δ tends to zero, but holds on a wide range of time scales.

Example : Monofractality of the log-volatility

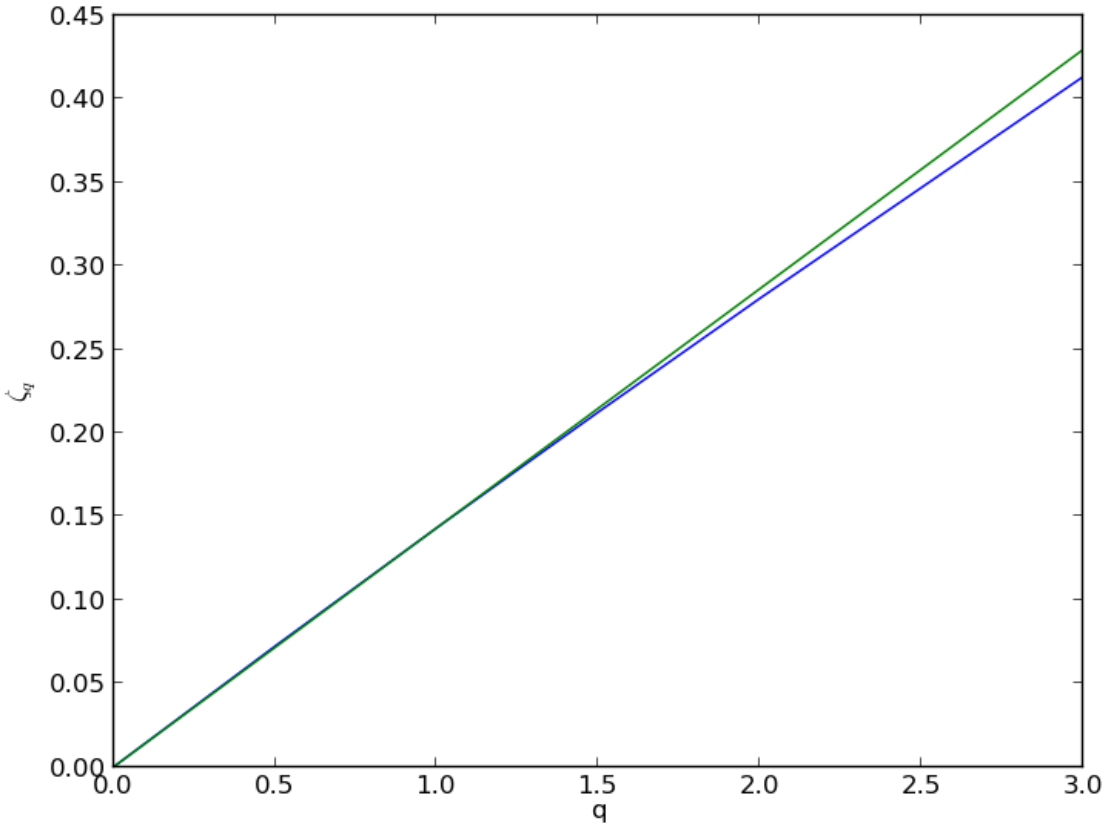


FIGURE : Empirical ζ_q and $q \to Hq$ with $H = 0.14$ (similar to a fBm with Hurst parameter H).

Modifying Heston model

Rough Heston model

It is natural to modify Heston model and consider its rough version :

$$dS_t = S_t \sqrt{V_t} dW_t$$

$$V_t = V_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda(\theta - V_s) ds + \frac{\lambda\nu}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sqrt{V_s} dB_s,$$

with $\langle dW_t, dB_t \rangle = \rho dt$ and $\alpha \in (1/2, 1)$.

Pricing in Heston models

Classical Heston model

From simple arguments based on the Markovian structure of the model and Ito's formula, we get that in the classical Heston model, the characteristic function of the log-price $X_t = \log(S_t/S_0)$ satisfies

$$\mathbb{E}[e^{iaX_t}] = \exp\big(g(a,t) + V_0 h(a,t)\big),$$

where h is solution of the following Riccati equation :

$$\partial_t h = \frac{1}{2}(-a^2 - ia) + \lambda(ia\rho\nu - 1)h(a,s) + \frac{(\lambda\nu)^2}{2}h^2(a,s), \quad h(a,0) = 0,$$

and

$$g(a,t) = \theta\lambda \int_0^t h(a,s)ds.$$

Pricing in Heston models (2)

Rough Heston models

Pricing in rough Heston models is much more intricate :

- Monte-Carlo : Bayer *et al.*, Bennedsen *et al.*
- Asymptotic formulas : Bayer *et al.*, Forde *et al.*, Fukasawa *et al.*, Jacquier *et al.*

This work

- Goal : Deriving a Heston like formula in the rough case.
- Tool : The microstructural foundations of rough volatility models based on Hawkes processes.
- We build a sequence of relevant high frequency models converging to our rough Heston process.
- We compute their characteristic function and pass to the limit.

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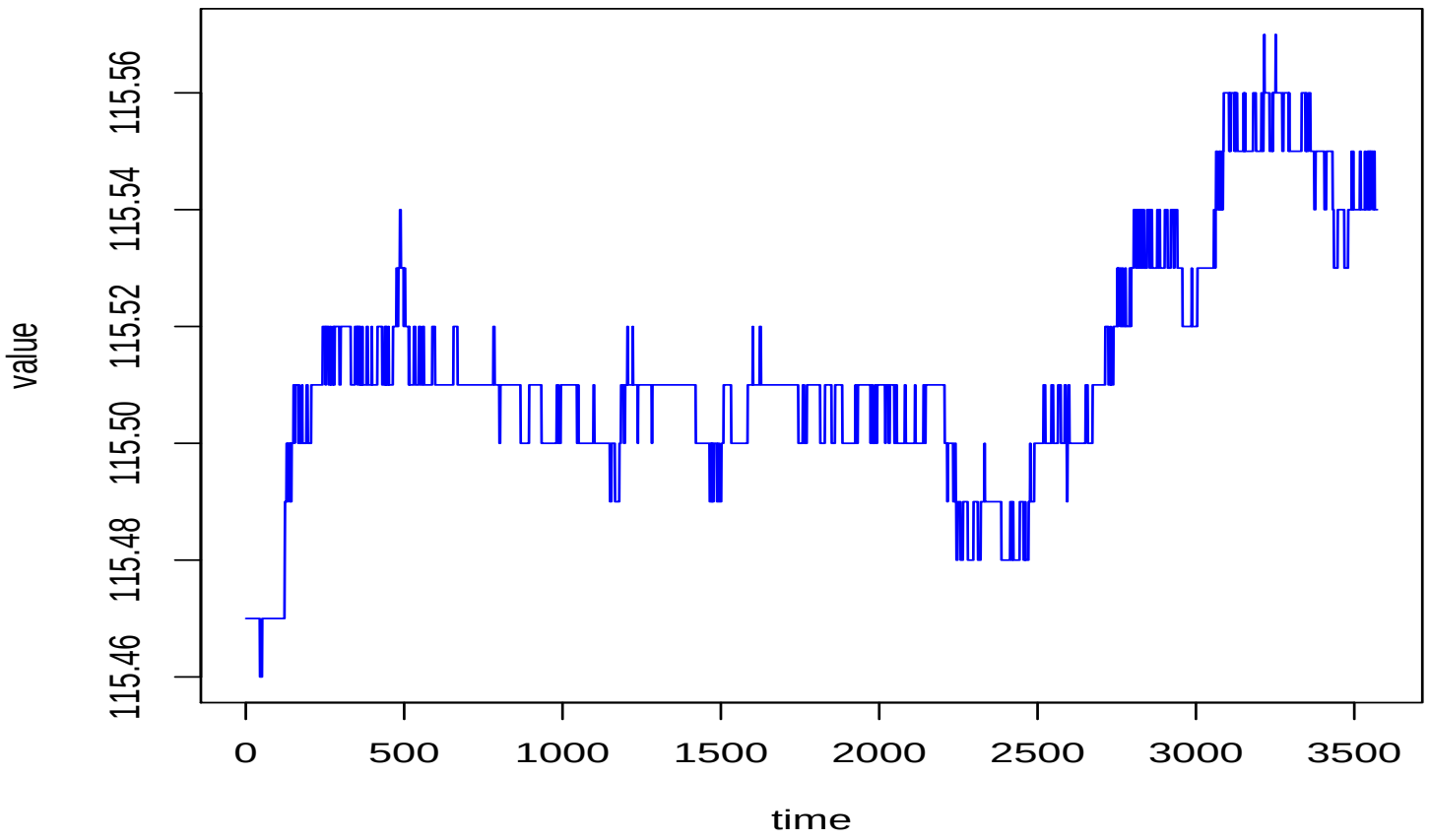
One financial asset, 1 hour



Building the model

Necessary conditions for a good microscopic price model

We want :

- A tick-by-tick model.
- A model reproducing the stylized facts of modern electronic markets in the context of high frequency trading.
- A model helping us to understand the rough dynamics of the volatility from the high frequency behavior of market participants.
- A model helping us to understand leverage effect.
- A model helping us to derive a Heston like formula.

Building the model

Stylized facts 1-2

- Markets are highly endogenous, meaning that most of the orders have no real economic motivations but are rather sent by algorithms in reaction to other orders, see Bouchaud *et al.*, Filimonov and Sornette.
- Mechanisms preventing statistical arbitrages take place on high frequency markets, meaning that at the high frequency scale, building strategies that are on average profitable is hardly possible.

Building the model

Stylized facts 3-4

- There is some asymmetry in the liquidity on the bid and ask sides of the order book. In particular, a market maker is likely to raise the price by less following a buy order than to lower the price following the same size sell order, see Brennan *et al.*, Brunnermeier and Pedersen, Hendershott and Seasholes.
- A large proportion of transactions is due to large orders, called metaorders, which are not executed at once but split in time.

Building the model

Hawkes processes

- Our tick-by-tick price model is based on Hawkes processes in dimension two, very much inspired by the approaches in Bacry *et al.* and Jaisson and R.
- A two-dimensional Hawkes process is a bivariate point process $(N_t^+, N_t^-)_{t\geq 0}$ taking values in $(\mathbb{R}^+)^2$ and with intensity $(\lambda_t^+, \lambda_t^-)$ of the form :

$$\begin{pmatrix} \lambda_t^+ \\ \lambda_t^- \end{pmatrix} = \begin{pmatrix} \mu^+ \\ \mu^- \end{pmatrix} + \int_0^t \begin{pmatrix} \varphi_1(t-s) & \varphi_3(t-s) \\ \varphi_2(t-s) & \varphi_4(t-s) \end{pmatrix} \cdot \begin{pmatrix} dN_s^+ \\ dN_s^- \end{pmatrix}.$$

Building the model

The microscopic price model

- Our model is simply given by

$$P_t = N_t^+ - N_t^- .$$

- N_t^+ corresponds to the number of upward jumps of the asset in the time interval $[0, t]$ and N_t^- to the number of downward jumps. Hence, the instantaneous probability to get an upward (downward) jump depends on the location in time of the past upward and downward jumps.
- By construction, the price process lives on a discrete grid.
- Statistical properties of this model have been studied in details.

Encoding the stylized facts

The right parametrization of the model

- Recall that

$$\begin{pmatrix}\lambda_t^+\\ \lambda_t^-\end{pmatrix} = \begin{pmatrix}\mu^+\\ \mu^-\end{pmatrix} + \int_0^t \begin{pmatrix}\varphi_1(t-s) & \varphi_3(t-s)\\ \varphi_2(t-s) & \varphi_4(t-s)\end{pmatrix} \cdot \begin{pmatrix}dN_s^+\\ dN_s^-\end{pmatrix}.$$

- High degree of endogeneity of the market$\rightarrow$ L^1 norm of the largest eigenvalue of the kernel matrix close to one.
- No arbitrage$\rightarrow$ $\varphi_1 + \varphi_3 = \varphi_2 + \varphi_4$.
- Liquidity asymmetry$\rightarrow$ $\varphi_3 = \beta\varphi_2$, with $\beta > 1$.
- Metaorders splitting$\rightarrow$ $\varphi_1(x),\ \varphi_2(x) \underset{x\to\infty}{\sim} K/x^{1+\alpha}$, $\alpha \approx 0.6$.

About the degree of endogeneity of the market

L^1 norm close to unity

- For simplicity, let us consider the case of a Hawkes process in dimension 1 with Poisson rate μ and kernel ϕ :

$$\lambda_t = \mu + \int_{(0,t)} \phi(t-s) dN_s.$$

- N_t then represents the number of transactions between time 0 and time t.
- L^1 norm of the largest eigenvalue close to unity$\rightarrow L^1$ norm of ϕ close to unity. This is systematically observed in practice, see Hardiman, Bercot and Bouchaud ; Filimonov and Sornette.
- The parameter $\|\phi\|_1$ corresponds to the so-called degree of endogeneity of the market.

About the degree of endogeneity of the market

Population interpretation of Hawkes processes

- Under the assumption $\|\phi\|_1 < 1$, Hawkes processes can be represented as a population process where migrants arrive according to a Poisson process with parameter μ.
- Then each migrant gives birth to children according to a non homogeneous Poisson process with intensity function ϕ, these children also giving birth to children according to the same non homogeneous Poisson process, see Hawkes (74).
- Now consider for example the classical case of buy (or sell) market orders. Then migrants can be seen as exogenous orders whereas children are viewed as orders triggered by other orders.

About the degree of endogeneity of the market

Degree of endogeneity of the market

- The parameter $\|\phi\|_1$ corresponds to the average number of children of an individual, $\|\phi\|_1^2$ to the average number of grandchildren of an individual,... Therefore, if we call cluster the descendants of a migrant, then the average size of a cluster is given by $\sum_{k\geq 1} \|\phi\|_1^k = \|\phi\|_1/(1-\|\phi\|_1)$.
- Thus, the average proportion of endogenously triggered events is $\|\phi\|_1/(1-\|\phi\|_1)$ divided by $1+\|\phi\|_1/(1-\|\phi\|_1)$, which is equal to $\|\phi\|_1$.

The scaling limit of the price model

Limit theorem

After suitable scaling in time and space, the long term limit of our price model satisfies the following rough Heston dynamics :

$$P_t = \int_0^t \sqrt{V_s} dW_s - \frac{1}{2} \int_0^t V_s ds,$$

$$V_t = V_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda(\theta - V_s) ds + \frac{\lambda\nu}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sqrt{V_s} dB_s,$$

with

$$d\langle W, B\rangle_t = \frac{1-\beta}{\sqrt{2(1+\beta^2)}} dt.$$

The scaling limit of the price model

Comments on the theorem

- The Hurst parameter $H = \alpha - 1/2$.
- Hence stylized facts of modern market microstructure naturally give rise to fractional dynamics and leverage effect.
- One of the only cases of scaling limit of a non ad hoc "micro model" where leverage effect appears in the limit. Compare with Nelson's limit of GARCH models for example.
- Uniqueness of the limiting solution is a difficult result. The proof requires the use of recent results in SPDEs theory by Mytnik and Salisbury.
- Obtaining a non-zero starting value for the volatility is a tricky point. To do so, we in fact consider a time-dependent μ.

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A general case

Multidimensional Hawkes process

- To obtain the characteristic function of our microscopic price process, we derive the characteristic function of multidimensional Hawkes processes.
- Let us consider a d-dimensional Hawkes process $N = (N^1, ..., N^d)$ with intensity

$$\lambda_t = \begin{pmatrix} \lambda_t^1 \\ \vdots \\ \lambda_t^d \end{pmatrix} = \mu(t) + \int_0^t \phi(t-s).dN_s.$$

Multidimensional Hawkes process

Population interpretation

- Migrants of type $k \in \{1, .., d\}$ arrive as a non-homogenous Poisson process with rate $\mu_k(t)$.
- Each migrant of type $k \in \{1, .., d\}$ gives birth to children of type $j \in \{1, .., d\}$ following a non-homogenous Poisson process with rate $\phi_{j,k}(t)$.
- Each child of type $k \in \{1, .., d\}$ also gives birth to other children of type $j \in \{1, .., d\}$ following a non-homogenous Poisson process with rate $\phi_{j,k}(t)$.

Multidimensional Hawkes process

Towards the characteristic function

- Let $(\tilde{N}^{k,j})_{1\leq j\leq d}$ be multivariate Hawkes processes with migrant rate $(\phi_{j,k})_{1\leq j\leq d}$ and kernel matrix ϕ.
- Let $N_t^{0,k}$ be the number of migrants of type k arrived up to time t of the initial Hawkes process.
- Let $T_1^k < ... < T^k_{N_t^{0,k}} \in [0,t]$ the arrival times of migrants of type k.
- We have

$$N_t^k \underset{law}{=} N_t^{0,k} + \sum_{1\leq j\leq d} \sum_{1\leq l\leq N_t^{0,j}} \tilde{N}^{j,k,(l)}_{t-T_l^j},$$

where the $(\tilde{N}^{j,k,(l)})$ are independent copies of $(\tilde{N}^{j,k})$.

Characteristic function of multidimensional Hawkes processes

Theorem

We have

$$\mathbb{E}[\exp(ia.N_t)] = \exp\Big(\int_0^t \big(C(a,t-s)-\mathbf{1}\big).\mu(s)ds\Big),$$

where $C : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{C}^d$ is solution of the following integral equation :

$$C(a,t) = \exp\big(ia + \int_0^t \phi^*(s).(C(a,t-s)-\mathbf{1})ds\big).$$

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Deriving the characteristic function

Strategy

- From our last theorem, we are able to derive the characteristic function of our high frequency price model.
- We then pass to the limit.

Characteristic function of rough Heston models

We write :

$$I^{1-\alpha}f(x) = \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{f(t)}{(x-t)^\alpha}dt, \; D^\alpha f(x) = \frac{d}{dx}I^{1-\alpha}f(x).$$

Theorem

The characteristic function at time t for the rough Heston model is given by

$$\exp\Big(\int_0^t g(a,s)ds + \frac{V_0}{\theta\lambda}I^{1-\alpha}g(a,t)\Big),$$

with $g(a,)$ the unique solution of the fractional Riccati equation :

$$D^\alpha g(a,s) = \frac{\lambda\theta}{2}(-a^2 - ia) + \lambda(ia\rho\nu - 1)g(a,s) + \frac{\lambda\nu^2}{2\theta}g^2(a,s).$$