Random walks in a 1D Levy random environment

Alessandra Bianchi
University of Padova

joint work with G.Cristadoro, M. Lenci, M. Ligabó
University of Bologna
Outline

1. MOTIVATIONS
   Anomalous diffusions

2. RELATED MODELS AND RESULTS
   Levy flights and walks and annealed results

3. MODEL
   1D Random walk in Levy random environment

4. RESULTS AND SOME IDEAS OF THE PROOF
   Quenched distribution and moments

5. CONCLUSIONS, WORKS IN PROGRESS AND OPEN PROBLEMS

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Motivations

**Anomalous diffusions**

Anomalous diffusions are stochastic processes $X(t) \in \mathbb{R}^d$ such that

$$
\mathbb{E}(X^2(t)) \sim t^\delta \quad \text{for } t \to \infty, \quad \delta \neq 1
$$

The behavior of superdiffusive processes ($\delta > 1$) characterizes many different natural systems and is mainly connected to

motion in disorder media:

- light particle in an optical lattice;
- tracer in a turbolent flow;
- efficient routing in network;
- predator hunting for food
Motivations

Main features

• long ballistic “flights“, where particle moves at constant velocity

• short disorder motion

Figure 1: Typical Levy flight
Related Models and results

**Models for anomalous diffusions**

**LEVY FLIGHTS**

Schlesinger, Klafter['85], Blumen, Klafter, Schlesinger, Zumofen ['90],

Random walk \((X(n))_{n \in \mathbb{N}}\) on \(\mathbb{R}^d\) with length steps given by a sequence of i.i.d. Levy \(\alpha\)-stable distribution with \(\alpha \in (0, 2)\):

\[ P(Z > x) \sim x^{-\alpha} \text{ for } x \to +\infty \]

\[ \text{Var}(Z) = +\infty \quad ; \quad \mathbb{E}(Z) \left\{ \begin{array}{ll}
< \infty & \text{if } \alpha \in (1, 2) \\
= \infty & \text{if } \alpha \in (0, 1]
\end{array} \right. \]
Models for anomalous diffusions

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heavy-tailed distribution \(\mathbb{P}(Z > x) \sim x^{-\alpha}\) for \(x \to +\infty\)

\[\longrightarrow \quad \text{Var}(Z) = +\infty \quad ; \quad \mathbb{E}(Z) \left\{ \begin{array}{l} < \infty \quad \text{if} \; \alpha \in (1, 2) \\ = \infty \quad \text{if} \; \alpha \in (0, 1) \end{array} \right.\]

Formally:

Given \((\xi_k)_{k \in \mathbb{N}}, \) i.i.d. \(U[S^{d-1}],\) independent of \((Z_k)_{k \in \mathbb{N}}, \) i.i.d Levy \(\alpha\)-stable

\[X(0) = 0 \quad , \quad X(n) = X(n - 1) + Z_n \xi_n, \quad n \geq 0\]
Related Models and results

**LEVY WALKS**

Stochastic process $(X(t))_{t \in \mathbb{R}^+}$ on $\mathbb{R}^d$ defined similarly to Levy flights but with jumps covered at constant velocity $v_0$. 
Related Models and results

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Formally:

Given \( (\xi_k)_{k \in \mathbb{N}}, \text{ i.i.d. } U[ S^{d-1} ], \) independent of \( (Z_k)_{k \in \mathbb{N}}, \text{ i.i.d Levy } \alpha \)-stable \( X(0) = 0 \) , \( X(t) = X\left(\frac{Z_k-1}{v_0}\right) + \xi_k v_0 t \) , for \( t \in \left(\frac{Z_k-1}{v_0}, \frac{Z_k}{v_0}\right] \).
Related Models and results

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\[X(0) = 0 \quad , \quad X(t) = X\left(\frac{Z_{k-1}}{v_0}\right) + \xi_k v_0 t \quad , \text{for} \ t \in \left(\frac{Z_{k-1}}{v_0}, \frac{Z_k}{v_0}\right)\]

Notice: in both processes **the lengths of the jumps (or inter-collision times) are independent**, 

\[\downarrow \quad \downarrow\]

scatterers are removed after each collision event.
Related Models and results

**Results on the second moments**

Levy flights and walks give rise to superdiffusive anomalous motion and in particular

$$
\mathbb{E}(X^2(t)) \sim \begin{cases} 
  t^2 & \text{if } \alpha \in (0, 1] \\
  t^{3-\alpha} & \text{if } \alpha \in (1, 2)
\end{cases} \quad \text{for } t \to \infty
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This suggests to model the transport in inhomogeneous material with the motion of a particle in a "Levy random environment".
Related Models and results

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**LEVY-LORENTZ GAS**

Barkai, Fleurov,Klafter[’00]

Motion of a particle in a fixed array of scatterers arranged randomly in such a way that the interdistances between them are i.i.d. $\alpha$-stable Levy random variables.
Model for a Lorentz-Levy gas

**MODEL**

**Levy Random environment**

- Let \((Z_k)_{k \in \mathbb{Z}}\) i.i.d. random variables taking value on \(\mathbb{N}^+\) and with law \(P\) s.t.

\[
P(Z > k) \sim k^{-\alpha} \quad \text{for } k \gg 1 \quad \text{(heavy tails)}
\]

- Construct a **Renewal Point Process on** \(\mathbb{Z}\), denoted by \(\text{PP}(Z) = \{\ldots Y_{-1} < Y_0 < Y_1 < \ldots\}\), s.t.

1. \(Y_0 = 0\)
2. \(|Y_k - Y_{k-1}| = Z_k\) so that \(Y_k = \text{sgn}(k) \sum_{j=1}^{|k|} Z_{\text{sgn}(k)j}, \quad k \neq 0\)

Levy Random environment \(\equiv \text{PP}(Z)\), i.e., scatterers are placed at points \(Y_k\).
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Model for a Lorentz-Levy gas

1D random walk in a Levy Random environment

- Let \((\xi_k)_{k \in \mathbb{N}}\) i.i.d. symmetric random variables taking value on \([-1, +1]\).

**Definition 1.** \(X(t), \ t \in \mathbb{N}\) is the process on \(\mathbb{Z}\) such that

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X(0) = 0
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\[
X(t + 1) = X(t) + \xi_{n(t)}, \text{ for } t > 0
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with \(n(t) = |\{s \leq t : X(s) \in \text{PP}(\mathbb{Z})\}| = \text{number of collisions up to } t.\)
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![Diagram of 1D random walk](image)
Model for a Lorentz-Levy gas

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\begin{align*}
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NOTE:

- Physically: Scatterers are now fixed by the environment and the increments have non trivial correlations.
Model for a Lorentz-Levy gas

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- **Physically:** Scatterers are now fixed by the environment and the increments have non-trivial correlations.
- **Mathematically:** The jump probabilities of the walk are now random themselves.

In particular, any given realization $z \in (\mathbb{N}^+)^\mathbb{Z}$ of the Levy environment, gives rise to distinguished jump probabilities $(p_{x,y}(z))_{x,y \in \mathbb{Z}}$. 
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For a fixed environment $z$, the law of $X(t)$ is called quenched law and denoted by $P_z$.

The law that comprises the entire randomness of the system, $\mathbb{P} = P \times P_z$, is called annealed law of the walk.
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Goal: Characterize the motion of \( X(t), t \in \mathbb{N} \).
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Goal: Characterize the motion of $X(t)$, $t \in \mathbb{N}$.

Remark: The random environment is not i.i.d., not even stationary, not elliptic, and the variance is infinite $\longrightarrow$ standard methods do not apply.
Previous (annealed) results

- For $\alpha \geq 2$ (finite variance) studied by Grassberger ('80), van Beijeren; Spohn ('83), Ernst; Dorfman; Nix; Jacobs ('95), Barkai; Fleurov ('99):
  (i) $\mathbb{E}(X^2(t)) \sim t$
  (ii) velocity autocorrelation function $\mathbb{E}(v(0)v(t)) \sim t^{-3/2}$. 
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- For $\alpha \in (1, 2)$ (infinite variance) Barkai, Fleuer, Klafter ['00] provide upper bounds on the (annealed) second moment:
  (i) $\mathbb{E}(X^2(t)) \geq c(\alpha)t^{2-\alpha}$ for $PP(Z)$ conditioned to contain 0
  (ii) $\mathbb{E}(X^2(t)) \geq c(\alpha)t^{3-\alpha}$ for stationary $PP(Z)$

where in stationary $PP(Z)$, $P(Y_1 = \ell) = \frac{\ell P(Z=\ell)}{E(Z)}$
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where in stationary PP($Z$), $P(Y_1 = \ell) = \frac{\ell P(Z=\ell)}{E(Z)}$

Main tools: Laplace transform and Tauberian theorem.

The result is compatible with a Levy flight scheme but not much informative for non-stationary initial conditions. Nothing is known about the quenched process.
For $n \in \mathbb{N}$, let $t(n) =$ time of the $n$th collision and set

$$\tilde{X}(n) \equiv X(t(n)), \quad n \in \mathbb{N}$$

- $\tilde{X}(n)$ is a SSRW on $\mathbb{P}(\mathbb{Z})$.
- Letting $S_n = \sum_{k=1}^{n} \xi_k$ the coupled SSRW on $\mathbb{Z}$, it holds

$$\tilde{X}(n) = Y_{S_n}$$

that is, $\tilde{X}(n)$ is the position of scatter label by $S_n$. 

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Model for a Lorentz-Levy gas
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\[ Y_1 \]

\[ Y_2 \]

\[ Z \]

\[ \text{PP}(Z) \]

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Model for a Lorentz-Levy gas

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Quenched law of $\tilde{X}(n)$

**Proposition 1.** For $\alpha \in (1, 2)$ and a $PP(Z)$ conditioned to contain 0, it holds

$$P_z \left( \frac{\tilde{X}(n)}{\mu \sqrt{n}} > x \right) \xrightarrow{n \to \infty} \int_x^{+\infty} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt \quad P\text{-a.s.}$$

where $\mu = E(Z_k)$. 
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where $\mu = \mathbb{E}(Z_k)$.

**Proof idea:** From $\tilde{X}(n) = Y_{S_n}$, we used

- CLT for $S_n$
- LLN for $Y_k$
Results and proofs ideas

**Quenched law of** \( X(t) \)

**THM 1.** For \( \alpha \in (1, 2) \) and a \( PP(Z) \) conditioned to contain 0, it holds

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**Quenched law of** \(X(t)\)

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\]

**Proof idea:** Write

\[
\frac{X(t)}{\sqrt{\mu t}} = \frac{X(t) - \hat{X}(n(t))}{\sqrt{\mu t}} + \frac{\hat{X}(n(t))}{\mu \sqrt{n(t)}} \sqrt{\mu \frac{n(t)}{t}}
\]

- \(\mathbb{E}_z \left( \left| \frac{X(t) - \hat{X}(n(t))}{\sqrt{\mu t}} \right| \right) \xrightarrow{t \to \infty} 0, \quad P - a.s\)
- By the ergodicity of the annealed process for the PVP

\[
\frac{n(t)}{t} \xrightarrow{t \to \infty} \frac{1}{\mu}, \quad \mathbb{P} - a.s
\]
Proposition 2. For $\alpha \in (1, 2)$ and a $\text{PP}(Z)$ conditioned to contain 0, it holds

$$E_z \left( \frac{\tilde{X}^m(n)}{n^{\frac{m}{2}}} \right) \xrightarrow{n \to \infty} \begin{cases} 0 & \text{for } m = 2k - 1 \\ \mu^m(m - 1)!! & \text{for } m = 2k \end{cases}, \quad P\text{-a.s.}$$

i.e., to the moments of $N(0, \mu^2)$. 
Quenched Moments of $\tilde{X}(n)$

**Proposition 2.** For $\alpha \in (1, 2)$ and a PP($Z$) conditioned to contain 0, it holds

$$E_z \left( \frac{\tilde{X}^m(n)}{n^{m/2}} \right) \xrightarrow{n \to \infty} \begin{cases} 0 & \text{for } m = 2k - 1 \\ \mu^m(m - 1)!! & \text{for } m = 2k \end{cases}, \quad P\text{-a.s.}$$

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**Proof idea:** From $\tilde{X}(n) = Y_{S_n}$, we used

- Moments convergence for $S_n$
- LLN for $Y_k$
Results and proofs ideas

**Quenched Moments of \( X(t) \)**

**THM 2.** *For \( \alpha \in (1, 2) \) and a PP(\( Z \)) conditioned to contain 0, it holds*

\[
E_z \left( \frac{X^m(t)}{t^{m/2}} \right) \xrightarrow{n \to \infty} \begin{cases} 
0 & \text{for } m = 2k - 1 \\
\frac{m}{\mu^2} (m - 1)!! & \text{for } m = 2k 
\end{cases}, \quad P\text{-a.s.}
\]

*i.e., to the moments of } \mathcal{N}(0, \mu).
Proof idea: Write $\frac{X^m(t)}{t^{m/2}} = \frac{X^m(t) - \tilde{X}^m(n(t))}{t^{m/2}} + \frac{\tilde{X}^m(n(t))}{n(t)^{m/2}} \left( \frac{n(t)}{t} \right)^{m/2}$

- For $\gamma > \frac{1}{2}$, define the event $E = \left\{ \left| X(t) \right| \wedge \left| \tilde{X}(n(t)) \right| < t^\gamma \right\}$

s.t. $P_z(E^c) \leq e^{-t^\gamma}$ \(P\text{-a.s}\)

Then

- $E_z \left( \left| \frac{X^m(t) - \tilde{X}^m(n(t))}{t^{m/2}} \right| \left| E^c \right\} \right) \leq 2t^{m/2} e^{-t^\gamma}$

- $E_z \left( \left| \frac{X^m(t) - \tilde{X}^m(n(t))}{t^{m/2}} \right| \left| E \right\} \right) \leq E_z \left( \left| X(t) - \tilde{X}(n(t)) \right| \left| E \right\} \right) \cdot mt^{\gamma(m-1)-m/2}$

Choosing $\frac{1}{2} < \gamma < \frac{m}{2(m-1)}$ and from $E_z \left( \left| X(t) - \tilde{X}(n(t)) \right| \right) \xrightarrow{t \to \infty} \mu$ we conclude.
Corollary 1. For $\alpha \in (1, 2)$ and a $PP(Z)$ conditioned to contain 0, it holds

$$\mathbb{E} \left( X^2(t) \right) \geq t, \quad \text{for } t \gg 1$$

This improves the annealed bound on the second moment given by BFK[00].
Results and proofs ideas

Conclusion, work in progress, open problems

• The quenched behavior of the 1 D Levy Lorentz gas with non-stationary initial condition do not displays anomalous diffusive behavior.

• Improved bound on the annealed second moment has been provided. Its exact behavior has still to be determined (work in progress).

• Under the stationary initial condition, we expect to find a similar behavior (work in progress)

• Study the model for \( \alpha \in (0, 1] \) (infinite-mean inter-collision times). A quenched super-diffusive behavior is conjectured (work in progress).

• Provide a similar construction for a 2 D Levy Lorentz gas (open problem).
Thank you for your attention!