Weak solutions of the Kolmogorov backward equations for option pricing in Lévy models

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joint work with Ernst Eberlein

Junior female researchers in probability
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Financial markets

- Stock market (Stocks, stock indices, stock and index options)
- Fixed income
- Commodity
- Foreign exchange
- Credit
- Other: Energy, weather, insurance...
Disciplines & Tasks

• Modeling
• Calibration
• Pricing
• Hedging
• Risk management of portfolios
• Portfolio optimization
Growing complexity of tools

1981 Calculator HP-12C of Hewlett-Packard: Black-Scholes formula implemented

1980s Big Bang
- Liberalization of European capital markets starting in UK
- Electronic trading: London Exchange, October 1986 (the trading floor was closed in 1988)

Today Algorithmic trading
"One uses them as sails for a happy voyage during a beneficent conjuncture and as an anchor of security in a storm."

de la Vega 1688 explains options in *Confusión de Confusiones*

Example: Pit inherited all Telecom stocks from his grandmother.

Put option: Gives the right to sell the stock in 1 year for 1 Euro.

What is the fair price of this option?
Principle of no arbitrage pricing

”There is no free lunch.”

Free lunch: Riskless gain without initial investment.

Any strategy

• with an initial cost \( \Pi \),
• no withdraw,
• with no further investments during the next year, and
• yielding the same terminal value as the put option in 1 year

must have initial cost \( \Pi = Put_0 \).

An arbitrage opportunity occurs if \( \Pi \neq Put_0 \).
Fair price of a put option

In a risk neutral world:  \( S_0 = E[e^{-r} S_1] \) and \( Put_0 = E[e^{-r} Put_1] \)

Risk-neutral pricing: Change to a risk-neutral probability measure.

Theorem (Fundamental Theorem of asset pricing)

Let the stock price be modeled on a finite probability space \((\Omega, \mathcal{F}, P)\). Then the following are equivalent:

(i) There is no arbitrage opportunity.

(ii) There exists a probability measure \( Q \) equivalent to \( P \) under which the discounted stock price is a martingale.
Black, Scholes, Merton

*Black Scholes model:* Stock price $S_t$ and riskless asset $r_t$ satisfy

$$S_t = S_0 e^{\sigma B_t + \mu t}, \quad r_t = e^{rt}$$

with standard Brownian motion $B$, drift $\mu$, volatility $\sigma > 0$, deterministic interest rate $r > 0$.

Under the risk-neutral probability measure:

$$S_t = S_0 e^{\sigma B_t + (r - \frac{\sigma^2}{2}) t}, \quad r_t = e^{rt}.$$ 

The model published in 1973 was awarded the Nobel prize in 1997.
Black & Scholes formula

**Black & Scholes model:** \( S_t = S_0 e^{\sigma B_t + (r - \frac{\sigma^2}{2}) t}, \ r_t = e^{rt} \).

The fair price of our put option is

\[
Put_0 = e^{-rT} E[\max\{1 - S_T, 0\}] = e^{-rT} E_x[g(\sigma B_T + (r - \frac{\sigma^2}{2}) T)]
\]

where \( B_0 = x := \log(S_0) \) under \( P_x \).

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Scaling property \( \sigma B_T \overset{d}{=} B_{\sigma^2 T} \)

\[
e^{rT} Put_0 = E_x[g(B_{\sigma^2 T} + (r - \frac{\sigma^2}{2}) T)] = U(\sigma^2 T, x + (r - \frac{\sigma^2}{2}) T)
\]

where \( U(t, x) = E_x[g(B_t)] \) solves the heat equation with initial condition \( g \).
Brownian motion

\( g : \mathbb{R} \to \mathbb{R} \) bounded and continuous, \( k : \mathbb{R} \to [0, \infty) \), \( K > 0 \) and \( 0 < a < \frac{1}{2T} \).

**Theorem (Feynman-Kac)**

Let \( u \in C^{1,2}((0, T) \times \mathbb{R}) \) with \( \max_{0 \leq t \leq T} |u(t, x)| \leq K e^{ax^2} \) solve

\[
\dot{u} + ku = \frac{1}{2} \Delta u \quad \text{on } (0, T] \times \mathbb{R}
\]

\[
u(0) = g \quad \text{on } \mathbb{R}.
\]

Then \( u \) admits the stochastic representation

\[
u(t, x) = E_x \left[ g(B_t) e^{-\int_0^t k(B_s) \, ds} \right].
\]
Gap between theory and practice

"In theory, theory and practice are the same. But in practice, they are very different."

Black & Scholes world:
- One parameter, one fair price
- Practically every claim is perfectly replicable

Market reality:
- More complex models
- Model uncertainty
Growing market complexity: Need for advanced techniques

"Das Signal an die Praxis des Risikomanagements ist jedenfalls klar: Sich nicht binden an ein einziges Modell, flexibel bleiben, die Modelle je nach Fragestellung variieren, immer mit Blick auf den "worst case"."

Hans Föllmer

Capturing more realistic features → higher complexity of the model

- Typically no closed form solution is available.
- Numerical methods are necessary.

A growing zoo of models with raising complexity makes a systematic study of related numerical methods necessary.
Figure: Dates: NASDAQ daily relative returns from 1971-April 2012, histogram in logarithmic scale.
Directly modeling of the log-returns

\[ S_t = S_0 e^{X_t}, \text{ for every } t \geq 0. \]

Consider log returns \((X_{t_n} - X_{t_{n-1}})\) iid but not any longer normally distributed as Black & Scholes assumed.

Lévy processes:

- Brownian motion
- Drift
- Jumps
Path of a Lévy process
### Definition

A stochastic process \((X_t)_{t \geq 0}\) with \(X_0 = 0\) is called a **Lévy process**, if

(L1) \(X\) has independent increments, i.e. \(X_t - X_s\) is independent of \(\mathcal{F}_s\) for every \(t \geq s\),

(L2) \(X\) has stationary increments, i.e. \(P_{X_{t+h} - X_t} = P_{X_h}\) for every \(h > 0\), and

(L3) \(X\) is stochastically continuous, i.e.

\[
\lim_{s \to t} P(|X_t - X_s| > \epsilon) = 0
\]

for every \(t \geq 0\) and \(\epsilon > 0\).
Lévy model

Model: \( S_t = S_0 e^{Lt}, \quad L_t = \sigma B_t + \mu t + L^d_t \)

- \( B \) Brownian motion, \( L^d \) pure jump process

Lévy-Khintchine formula: \( E e^{iuL_t} = e^{-tA(-u)} \) with symbol

\[
A(u) = \frac{\sigma^2}{2} u^2 + i bu - \int (e^{-iuy} - 1 + iuy) F(dy)
\]

- Approach to Lévy models: via Fourier transform
\[ \nabla_t^g = E[g(L_T) | \mathcal{F}_t] = \left. \frac{E[g(L_{T-t} + x)]}{x=L_t} \right|_{x=L_t} =: \left. \Gamma_{T-t} g(x) \right|_{x=L_t} = u(t,x) \]

\[ \partial_t u(t, x) = \lim_{h \downarrow 0} \frac{\Gamma_h - 1}{h} u(t, x) = -G u(t, x) \]

by the definition of the inf. generator \( G \) of \( L \) i.e.

\[ \partial_t u + G u = 0 , \quad u(T, x) = g(x) \]
PIDE for option pricing

$$\Pi_t^g = E[g(L_{T-t} + x)] \bigg|_{x=L_t} =: u(t, L_t)$$

$$\partial_t u + G u = 0, \quad u(T, x) = g(x)$$

with infinitesimal generator $G = \frac{\sigma^2}{2} \frac{d^2}{dx^2} + \mu \frac{d}{dx} + \mathcal{G}^{\text{jump}}$

$$G_{\text{jump}} \varphi(x) = \int \left( \varphi(x + y) - \varphi(x) - h(y) \varphi'(x) \right) F(dy)$$
PDE methods for pricing

- Pricing path-dependent options: E.g. American and barrier options.
- Option pricing beyond Lévy models.

Arsenal of Galerkin methods:

- Standard Finite Element (Achdou, Pironneau, Schwab, etc.)
- Wavelet-Galerkin (Schwab)
- Reduced Basis (Cont, Pironneau).
Outline

We pose the following questions:

(1) *Under which conditions on the symbol is the Kolmogorov equation parabolic? With tractable solution space?*
   Sobolev-Slobodeckiđ space, Sobolev index of a Lévy process

(2) *How to adapt the concept for realistic option payoffs?*
   Exponentially weighted spaces

(3) *Does the variational solution yield the option price?*
   Feynman-Kac formula for variational solutions
Galerkin Method

Gelfand triplet $V \hookrightarrow H \hookrightarrow V^*$ and $A : V \to V^*$ linear with bilinear form

$$a(u, v) := (A u)(v) = \langle A u, v \rangle_{V^* \times V}.$$ 

Continuity

$$a(u, w) \leq C_1 \|u\|_V \|w\|_V \quad \forall u, w \in V,$$

Gårding inequality

$$a(u, u) \geq C_2 \|u\|^2_V - C_3 \|u\|^2_H \quad \forall u \in V,$$

with $C_1, C_2 > 0$ and $C_3 \geq 0$. 
Variational (or weak) solutions of parabolic equations:

**Theorem**

For \( f \in L^2(0, T; V^*) \) and \( g \in H \),

\[
\partial_t u + A u = f \\
u(0) = g,
\]

has a unique solution \( u \in W^1 \), i.e. \( u \in L^2(0, T; V) \) with \( \dot{u} \in L^2(0, T; V^*) \).
Sobolev index of a Lévy process

Fourier transform of $Au$:

A Pseudo Differential Operator $\mathcal{F}(Au) = A\mathcal{F}(u)$

Symbol $A(\xi) = \frac{\sigma^2}{2} \xi^2 + ib\xi - \int (e^{-i\xi y} - 1 - i\xi h(y)) F(dy)$
Sobolev index of a Lévy process

Bilinear form via Symbol for \( u, v \in C_0^\infty \)

\[
a(u, v) = \int (Au)v = \frac{1}{2\pi} \int A\hat{u}\hat{v} \quad \text{(Parseval)}
\]

Sobolev-Slobodeckii space \( H^s := \{ u \in L^2 \mid \|u\|_{H^s} < \infty \} \) with norm

\[
\|u\|_{H^s}^2 = \int |\hat{u}(\xi)|^2 (1 + |\xi|^s)^2 \, d\xi
\]
Sobolev index of a Lévy process

Definition

Let $A$ be a symbol (and $L$ be a Lévy process with symbol $A$). We call $\alpha \in (0, 2]$ the Sobolev index of the symbol $A$ (resp. of the process $L$), if

- Continuity condition: $|A(\xi)| \leq C_1 (1 + |\xi|)^{\alpha}$
- Gårding condition: $\Re(A(\xi)) \geq C_2 (1 + |\xi|)^{\alpha} - C_3 (1 + |\xi|)^{\beta}$

for $\xi \in \mathbb{R}$ with $C_1, C_2 > 0$ and $C_3 \geq 0$, $0 \leq \beta < \alpha$. 

Kathrin Glau (TUM), Junior female researchers in probability, Berlin 11.10.2013
Sobolev index of a Lévy process

\[ \dot{u} + A u = f, \quad u(0) = g \]

(1)

with \( f \in L^2(0, T; H^{-s}(\mathbb{R}^d)) \) and initial condition \( g \in L^2(\mathbb{R}^d) \).

**Theorem**

Let \( A \) be a PDO with symbol \( A \) and bilinear form \( a \).

Then (i) \( \implies \) (ii). If additionally \( A \) is continuous, (i) \( \iff \) (ii).

(i) The symbol \( A \) has a Sobolev index \( \alpha \in (0, 2] \).

(ii) The evolution problem (1) is parabolic w.r.t. Gelfand triplet \( (H^{\alpha/2}(\mathbb{R}^d), L^2(\mathbb{R}^d), H^{-\alpha/2}(\mathbb{R}^d)) \) and has a unique variational solution \( u \in W^1(0, T; H^{\alpha/2}(\mathbb{R}^d), L^2(\mathbb{R}^d)) \).
Sobolev index measures smoothing

**Proposition**

If the Lévy process has Sobolev index $\alpha$, then for every $t > 0$, $P^L_t$ has a smooth and bounded Lebesgue density.

Let $\tilde{H}^{\alpha/2}(D) := \{ u \in H^{\alpha/2}(\mathbb{R}^d) | u|_{D^c} = 0 \}$.

Set $a := a|_{\tilde{H}^{\alpha/2}(D)}\times\tilde{H}^{\alpha/2}(D)$.

**Corollary**

Let $\mathcal{A}$ be a PDO with symbol $A$ and Sobolev index $\alpha$. Let $D \subset \mathbb{R}^d$ be open with the segment property. Then

\[
\partial_t u + \mathcal{A} u = f \\
u(0) = g,
\]

for $f \in L^2(0, T; \tilde{H}^{-\alpha/2}(D))$ and initial condition $g \in L^2(D)$ has a unique weak solution $u$ in the space $W^1(0, T; \tilde{H}^{\alpha/2}(D), L^2(D))$. 
For a Lévy process with characteristics \((b, c, F)\),

\[
\beta := \inf \left\{ \gamma > 0 \left| \int_{[-1,1]} |x|^\gamma F(dx) < \infty \right. \right\}
\]

is called the Blumenthal-Getoor index of the process.

**Proposition**

Let \(L\) be a Lévy process with characteristics \((b, 0, F)\) and Blumenthal-Getoor index \(\beta\).

(a) If \(\beta < 1\) or \(\beta = 1\) and \(\int_{-1}^{1} |x|F(dx) < \infty\),
then \(P\text{-a.e. path is of bounded variation on } (0, t] \ \forall t > 0\).

(b) If \(\beta > 1\) or \(\beta = 1\) and \(\int_{-1}^{1} |x|F(dx) = \infty\),
then \(P\text{-a.e. path is of unbounded variation on } (0, t] \ \forall t > 0\).
Sobolev index and Blumenthal-Getoor index

**Theorem**

For an $\mathbb{R}$-valued Lévy process with characteristics $(b, 0, F)$ and Sobolev index $0 < \alpha < 2$, $\alpha$ equals the Blumenthal-Getoor index.

**Question:** Given $L$ has Blumenthal-Getoor index $\beta$ find sufficient conditions such that $\beta$ is the Sobolev index.
Sobolev index and Blumenthal-Getoor index

Example

Let \( L \) be a Lévy process with characteristics \((0, 0, F)\) with Lévy density \( F(dx) := -|x|^{-1-\alpha} \log(|x|)1_{[-1,1]}(x) \, dx \). Then

\[
\int_{-1}^{1} |x|^{\alpha-\epsilon} F(dx) = \infty, \quad \int_{-1}^{1} |x|^{\alpha+\epsilon} F(dx) < \infty.
\]

I.e. \( \alpha \) is the Blumenthal-Getoor index of \( L \). Moreover,

\[
A(u) \geq |u|^{\alpha} \log(|u|) \int_{-|u|}^{|--u|} (1 - \cos(y))|y|^{-1-\alpha} \, dy,
\]

hence the continuity condition is not satisfied with index \( \alpha \).
Proposition

Let $L$ be an $\mathbb{R}$-valued Lévy process with characteristics $(b, 0, F)$. Let $0 < \beta < 2$.

(i) If $\int_{[-1,1]} |x|^{\beta} F(dx) < \infty$ and $|\mathfrak{S}(A(u))| \leq C(1 + |u|^{\beta})$, then the continuity condition is satisfied with index $\alpha := \beta$.

(ii) If $\liminf_{r \downarrow 0} r^{\alpha - 2} \int_{[-r,r]} |x|^{2} F(dx) > 0$, then the Gårding condition is satisfied with index $\alpha := \beta$. 
Sobolev index: Examples

Example

(i) Every pure jump Lévy process which has a Sobolev index has infinite jump activity, i.e. $F(\mathbb{R}) = \infty$.

(ii) For every Lévy process with Sobolev index $\alpha > 1$, we have
$$\int_{[-1,1]} |x| F(dx) = \infty.$$  

(iii) Compound Poisson processes do not have a Sobolev index.

(iv) Variance gamma processes do not have a Sobolev index.
Example (Multivariate $\alpha$-semi-stable Lévy process)

Let $L$ be a nondegenerate $\mathbb{R}^d$-valued $\alpha$-semi-stable Lévy process.

a) If $0 < \alpha < 1$, then $L$ has Sobolev index $\alpha$ if and only if $L$ is strictly $\alpha$-semi-stable.

b) If $1 < \alpha \leq 2$, then $L$ has Sobolev index $\alpha$.

Let $L \neq 0$ be an $\mathbb{R}$-valued 1-stable Lévy process.

c) $L$ has Sobolev index 1 if and only if $L$ is strictly 1-stable.
## Sobolev index: Examples

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Sobolev index $\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brownian motion (+ drift) (+jumps)</td>
<td>2</td>
</tr>
<tr>
<td>GH, NIG</td>
<td>1</td>
</tr>
<tr>
<td>CGMY, no drift</td>
<td>$Y$</td>
</tr>
<tr>
<td>CGMY $Y \geq 1$</td>
<td>$Y$</td>
</tr>
<tr>
<td>Student-$t$</td>
<td>1</td>
</tr>
<tr>
<td>Cauchy</td>
<td>1</td>
</tr>
<tr>
<td>Variance Gamma</td>
<td>–</td>
</tr>
</tbody>
</table>
Sobolev index of a Lévy process

Figure: Values of experimental order of convergence, $EOC(l)$ for different values of $Y$ and different levels $l$ with fixed level $L = 14$.

Figure: Average $\frac{1}{9} \sum_{l=3}^{11} EOC(l)$ for different values of $Y$.
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(1) Under which conditions on the symbol is the Kolmogorov equation parabolic? With tractable solution space?
   - Glau, Classification of Lévy processes with parabolic Kolmogorov backward equations, submitted

(2) How to adapt the concept for realistic option payoffs?
   Exponentially weighted spaces

(3) Does the variational solution yield the option price?
   Feynman-Kac formula for variational solutions
$L^2$-based approach: Initial condition $g \in L^2$

but $(S_T - K)^+ = (e^{LT} - K)^+ = g(L_T) \implies g \notin L^2$

exponential weighting

$g \in L^2_\eta \iff g_\eta = g e^{\langle \eta, \cdot \rangle} \in L^2$

weighted Sobolev spaces $H^s_\eta$

$$\|u\|_{H^s_\eta} = \|u_\eta\|_{H^s} = \int |\mathcal{F}(u_\eta)(\xi)|^2 (1 + |\xi|)^{2s} \, d\xi$$
PIDE and option pricing

exponentially weighted spaces

\[ \Rightarrow \]

shifting the symbol in the complex plane
Let $\eta \in \mathbb{R}^d$ with

(A1) **Exponential moments** $\int_{|x|>1} |e^{\langle z, x \rangle} |F(dx) < \infty$, $\forall z \in U_\eta$.

(A2) **Continuity** $|A(z)| \leq C_1 (1 + |z|)^\alpha$.

(A3) **Gårding** $\Re(A(z)) \geq C_2 (1 + |z|)^\alpha - C_3 (1 + |z|)^\beta$.

For all $z \in U_\eta$ with constants $C_1, C_2 > 0$, $C_3 \geq 0$ and $0 \leq \beta < \alpha$. 

Kathrin Glau (TUM), Junior female researchers in probability, Berlin 11.10.2013
Theorem

Let (A1)–(A3) with $\alpha \geq 1$ and $\eta > 0$. Then $\exists! \ u \in W^1$ which is the weak solution of

$$\partial_t u + A u = 0 \quad \text{in } (H^{\alpha/2}_\eta)^*$$

$$u(0) = g.$$
We pose the following questions:

(1) Under which conditions on the symbol is the Kolmogorov equation parabolic? With tractable solution space?
   - Glau, *Classification of Lévy processes with parabolic Kolmogorov backward equations*, submitted

(2) How to adapt the concept for realistic option payoffs?
   - Eberlein and Glau, *Variational solutions of the pricing PIDEs for European options in Lévy models*, to appear in Applied Mathematical Finance

(3) Does the variational solution yield the option price?
   - Feynman-Kac formula for variational solutions
Feynman-Kac formula for processes with killing rate

- Now we generalize from Lévy processes to time-inhomogeneous processes and PIJAC.

Consider the generalized Kolmogorov backward equation

\[
\partial_t u + \mathcal{A}_{T-t} u + \kappa_{T-t} u = f, \quad u(0) = g,
\]

with a bounded killing rate \( \kappa \).

Question: Under which conditions \( \exists! \) weak solution with stochastic representation

\[
u(T - t, L_t) = E\left(g(L_T) e^{-\int_t^T \kappa_h(L_{h^-}) \, dh} \right. \\
+ \left. \int_t^T f(T - s, L_s) e^{-\int_s^t \kappa_h(L_{h^-}) \, dh} \, ds \bigg| \mathcal{F}_t \right)
\]
Assumptions

Let $L$ be an $\mathbb{R}^d$-valued PIIAC and let $A : (t, z) \rightarrow A_t(z)$, $A : [0, T] \times U_{-\eta} \rightarrow \mathbb{C}$ and fix some $\alpha > 0$ and some $\eta \in \mathbb{R}^d$.

- $d$-dimensional: $U_{-\eta} = U_{-\eta^1} \times \cdots \times U_{-\eta^d}$
- Define $R_{\eta}$ as cartesian product of imaginary parts of $U_{-\eta}$.
Assumptions

(A1) For the Lévy measure $F$ of $L$ we have for all $\eta' \in R_\eta$.

$$
\int_0^T \int_{|x|>1} e^{2|\langle \eta', x \rangle|} F_s(dx) \, ds < \infty. \quad \text{(exponential moments)}
$$

(A2) There exists a constant $C_1 > 0$ with

$$
|A_t(\xi + i\eta')| \leq C_1 (1 + |\xi|)^\alpha \quad \text{(continuity condition)}
$$

for all $\xi \in \mathbb{R}^d$, all $\eta' \in R_\eta$ and for all $t \in [0, T]$.

(A3) There exist constants $C_2 > 0$ and $C_3 \geq 0$, such that for a $0 \leq \beta < \alpha$

$$
\Re(A_t(\xi + i\eta')) \geq C_2 (1 + |\xi|)^\alpha - C_3 (1 + |\xi|)^\beta \quad \text{(Gårding condition)}
$$

for all $\xi \in \mathbb{R}^d$, all $\eta' \in R_\eta$ and for all $t \in [0, T]$.

(A4) The mapping $t \mapsto A_t(\xi - i\eta)$ is continuous for every fixed $\xi \in \mathbb{R}^d$.

(A5) The mapping $t \mapsto \Re(A_t(\xi - i\eta))$ is continuously differentiable for every fixed $\xi \in \mathbb{R}^d$ with

$$
|\partial_t \Re(A_t(\xi - i\eta))| \leq C_4 (1 + |\xi|)^\alpha \quad \text{for all } \xi \in \mathbb{R}^d.
$$
Theorem (Feynman-Kac Representation)

Let $L$ be an $\mathbb{R}^d$-valued PIIAC satisfying assumptions (A1)–(A5). Let $f \in L^2(0, T; H^1_\eta(\mathbb{R}^d))$, $\kappa : [0, T] \times \mathbb{R}^d \to [0, \infty)$ be bounded such that $t \mapsto \kappa_t(x)$ is continuous for a.e. $x \in \mathbb{R}^d$, and $g \in L^2_\eta(\mathbb{R}^d)$.

Then the unique weak solution $u \in W^1$ of

$$\partial_t u + A_{T-t} u + \kappa_{T-t} u = f, \quad u(0) = g,$$

has for every $t \in (0, T]$ the stochastic representation

$$u(T-t, L_t) = E \left( g(L_T) \, e^{-\int_t^T \kappa_h(L_{h^-}) \, dh} \right)$$

$$+ \int_t^T f(T-s, L_s) \, e^{-\int_t^s \kappa_h(L_{h^-}) \, dh} \, ds \left| \mathcal{F}_t \right) \quad \text{a.s.}$$
Proof of Feynman-Kac Representation

**Sketch of proof:**

- Continuity and Gårding condition for $\mathcal{A} + \kappa$ are trivial.
- A regularity assertion implies

$$\dot{u} + \mathcal{A} u + \kappa u = f \quad \text{a. e.}$$

For $w(t, x) := u(T - t, x)$ we have

$$\int [\dot{w} + \mathcal{A}_s w + \kappa w](s, L_s) \, ds = \int -f(T - s, L_s) \, ds$$
Proof of Feynman-Kac Representation

- Itô requires higher regularity, so approximate $w$ by a smooth sequence. Then:

$$w^n(s, L_s) = E \left( w^n(t, L_t) e^{-\int_s^t \kappa_h(L_h) \, dh} \right)$$

$$+ E \left( \int_s^t \left( \dot{w}^n - A_h w^n - \kappa w^n \right)(h, L_{h-}) e^{-\int_s^h \kappa \lambda (L_{\lambda-}) \, d\lambda} \, dh \bigg| \mathcal{F}_s \right).$$

Taking the limit $n \to \infty$ yields

$$w(s, L_s) = E \left( w(t, L_t) e^{-\int_s^t \kappa_h(L_h) \, dh} + \int_s^t f(T - h, L_{h-}) e^{-\int_s^h \kappa \lambda (L_{\lambda-}) \, d\lambda} \, dh \bigg| \mathcal{F}_s \right).$$
Hölder continuity

Corollary

We have for a.e. \( x \in \mathbb{R}^d \)

\[
 u(T, x) = E_x \left( g(L_T) e^{-\int_0^T \kappa_h(L_{h^-}) dh} + \int_0^T f(T-s, L_s) e^{-\int_0^s \kappa_h(L_{h^-}) dh} ds \right).
\]

- For \( \alpha \in (1, 2] \) and \( d = 1 \), \( x \mapsto u(t, x) \) is \( \lambda \)-Hölder continuous with \( \lambda = \frac{\alpha-1}{2} \) for any fixed \( t \in (0, T) \).
- In particular, \( x \mapsto u(t, x) \) is continuous and the expression for \( u(T, x) \) holds for every \( x \in \mathbb{R} \).

\[ \varphi : \mathbb{R}^d \to \mathbb{R} \] is \( \lambda \)-Hölder continuous iff

\[
 \sup_{x,y \in \mathbb{R}, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{\lambda}} < \infty.
\]
Applications

• For killing rates of indicator type, \( \kappa := \lambda 1_D \) for \( D \subset \mathbb{R}^d \) and \( \lambda > 0 \), initial condition \( g \equiv 1 \), \( f \equiv 0 \) and \( L_0 = x \), we get

\[
u(T, x) = E_x \left( e^{-\lambda \int_0^T 1_D (L_h^-) \, dh} \right),
\]

which is the \textbf{Laplace transform} at \( \lambda \) of the \textbf{occupation time} \( \int_0^T 1_D (L_h^-) \, dh \) of \( L \) spent until time \( T \) in the domain \( D \).

• \textbf{Ruin theory}: Replacing concept of ruin by bankruptcy, see
  • Albrecher and Lautscham (2013) and
  • Albrecher, Gerber and Shiu (2011).

• Employee stock options,

• relativistic quantum mechanics, see
(1) **Under which conditions on the symbol is the Kolmogorov equation parabolic? With tractable solution space?**

- Glau, *Classification of Lévy processes with parabolic Kolmogorov backward equations*, submitted

(2) **How to adapt the concept for realistic option payoffs?**

- Eberlein and Glau, *Variational solutions of the pricing PIDEs for European options in Lévy models*, to appear in Applied Mathematical Finance

(3) **Does the variational solution yield the option price?**

- Glau, *Feynman-Kac formula for Lévy processes with killing rate and bankruptcy*, working paper
- Eberlein and Glau, *Feynman-Kac theorem for Lévy processes and weak solutions to boundary value problems*, work in prog.
- Glau, *Feynman-Kac-Darstellung zur Optionspreisbewertung in Lévy-Modellen*, PhD-Thesis

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Thank you for your attention!