Weak solutions of the Kolmogorov backward equations for option pricing in Lévy models

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joint work with Ernst Eberlein

Junior female researchers in probability
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Financial markets

- Stock market (Stocks, stock indices, stock and index options)
- Fixed income
- Commodity
- Foreign exchange
- Credit
- Other: Energy, weather, insurance...

Disciplines & Tasks

- Modeling
- Calibration
- Pricing
- Hedging
- Risk management of portfolios
- Portfolio optimization

Growing complexity of tools

1981 Calculator HP-12C of Hewlett-Packard: Black-Scholes formula implemented

1980s Big Bang

- Liberalization of European capital markets starting in UK
- Electronic trading: London Exchange, October 1986 (the trading floor was closed in 1988)

Today Algorithmic trading

Fair price of derivatives

"One uses them as sails for a happy voyage during a beneficent conjuncture and as an anchor of security in a storm."

de la Vega 1688 explains options in *Confusión de Confusiones*

Example: Pit inherited all Telecom stocks from his grandmother.

Put option: Gives the right to sell the stock in 1 year for 1 Euro.

What is the fair price of this option?

Principle of no arbitrage pricing

"There is no free lunch."

Free lunch: Riskless gain without initial investment.

Any strategy

- with an initial cost Π,
- no withdraw,
- with no further investments during the next year, and
- yielding the same terminal value as the put option in 1 year

must have initial cost $\Pi = Put_0$.

An arbitrage opportunity occurs if $\Pi \neq Put_0$.

Fair price of a put option

In a risk neutral world: $S_0 = E[\mathrm{e}^{-r} S_1]$ and $Put_0 = E[\mathrm{e}^{-r} Put_1]$

Risk-neutral pricing: Change to a risk-neutral probability measure.

Theorem (Fundamental Theorem of asset pricing)

Let the stock price be modeled on a finite probability space $(\Omega, \mathcal{F}, P)$. Then the following are equivalent:

(i) *There is no arbitrage opportunity.*

(ii) *There exists a probability measure Q equivalent to P under which the discounted stock price is a martingale.*

Black, Scholes, Merton

Black Scholes model: Stock price S_t and riskless asset r_t satisfy

$$S_t = S_0 \, \mathrm{e}^{\sigma B_t + \mu t}, \quad r_t = \mathrm{e}^{rt}$$

with standard Brownian motion B, drift μ, volatility $\sigma > 0$, deterministic interest rate $r > 0$.

Under the risk-neutral probability measure:

$$S_t = S_0 \, \mathrm{e}^{\sigma B_t + (r - \frac{\sigma^2}{2})t}, \quad r_t = \mathrm{e}^{rt} \, .$$

The model published in 1973 was awarded the Nobel prize in 1997.

Black & Scholes formula

Black & Scholes model: $S_t = S_0\, \mathrm{e}^{\sigma B_t + (r - \frac{\sigma^2}{2})t}$, $r_t = \mathrm{e}^{rt}$.
The fair price of our put option is

$$Put_0 = \mathrm{e}^{-rT}\, E[\max\{1 - S_T, 0\}] = \mathrm{e}^{-rT}\, E_x\big[g(\sigma B_T + \big(r - \frac{\sigma^2}{2}\big)\, T)\big]$$

where $B_0 = x := \log(S_0)$ under P_x.

Scaling property $\sigma B_T \stackrel{d}{=} B_{\sigma^2 T}$

$$\mathrm{e}^{rT}\, Put_0 = E_x\big[g(B_{\sigma^2 T} + \big(r - \frac{\sigma^2}{2}\big)\, T\big] = U(\sigma^2 T, x + \big(r - \frac{\sigma^2}{2}\big)\, T]$$

where $U(t,x) = E_x\Big[g(B_t)\Big]$ solves the heat equation with initial condition g.

Brownian motion

$g : \mathbb{R} \to \mathbb{R}$ bounded and continuous, $k : \mathbb{R} \to [0, \infty)$,
$K > 0$ and $0 < a < \frac{1}{2T}$.

Theorem (Feynman-Kac)

Let $u \in C^{1,2}((0,T) \times \mathbb{R})$ *with* $\max_{0 \le t \le T} |u(t,x)| \le K\, \mathrm{e}^{ax^2}$ *solve*

$$\begin{aligned} \dot{u} + ku &= \frac{1}{2}\Delta u && \text{on } (0,T] \times \mathbb{R} \\ u(0) &= g && \text{on } \mathbb{R}. \end{aligned}$$

Then u *admits the stochastic representation*

$$u(t,x) = E_x\left[g(B_t)\, \mathrm{e}^{-\int_0^t k(B_s)\,\mathrm{d}s} \right].$$

Gap between theory and practice

"In theory, theory and practice are the same. But in practice, they are very different."

Black & Scholes world:

- One parameter, one fair price
- Practically every claim is perfectly replicable

Market reality:

- More complex models
- Model uncertainty

Growing market complexity: Need for advanced techniques

"Das Signal an die Praxis des Risikomanagements ist jedenfalls klar: Sich nicht binden an ein einziges Modell, flexibel bleiben, die Modelle je nach Fragestellung variieren, immer mit Blick auf den "worst case"."
Hans Föllmer

Capturing more realistic features $\Longrightarrow$ higher complexity of the model

- Typically no closed form solution is available.
- Numerical methods are necessary.

A growing zoo of models with raising complexity makes a systematic study of related numerical methods necessary.

Lévy processes

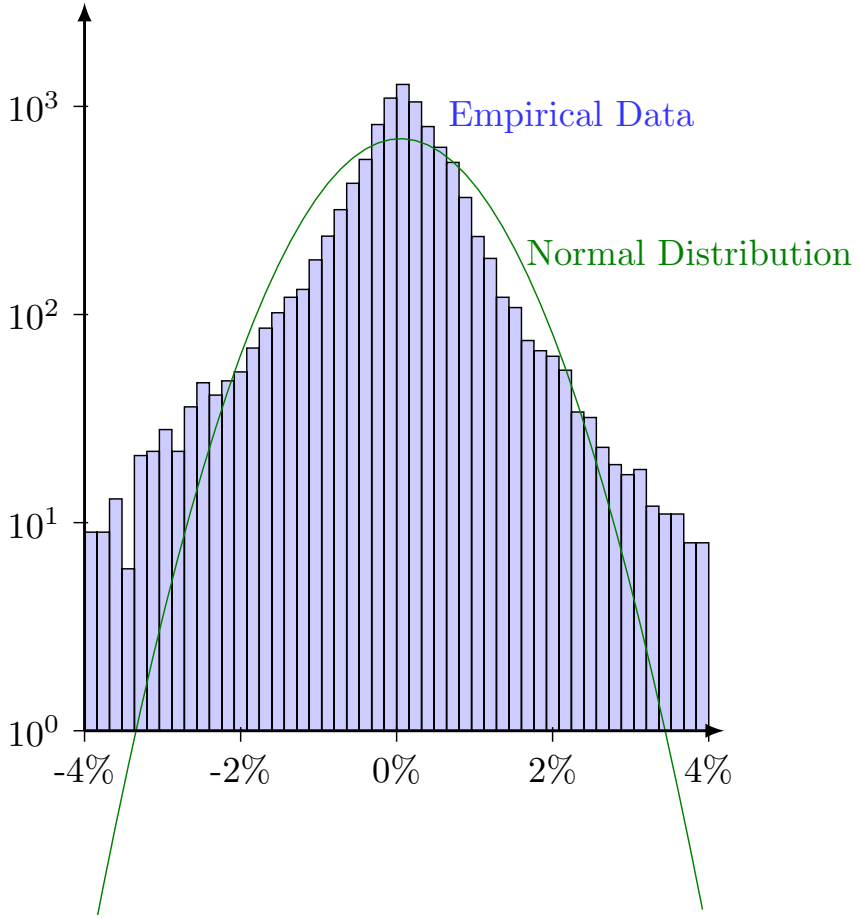


Figure: Dates: NASDAQ daily relative returns from 1971-April 2012, histogram in logarithmic scale.

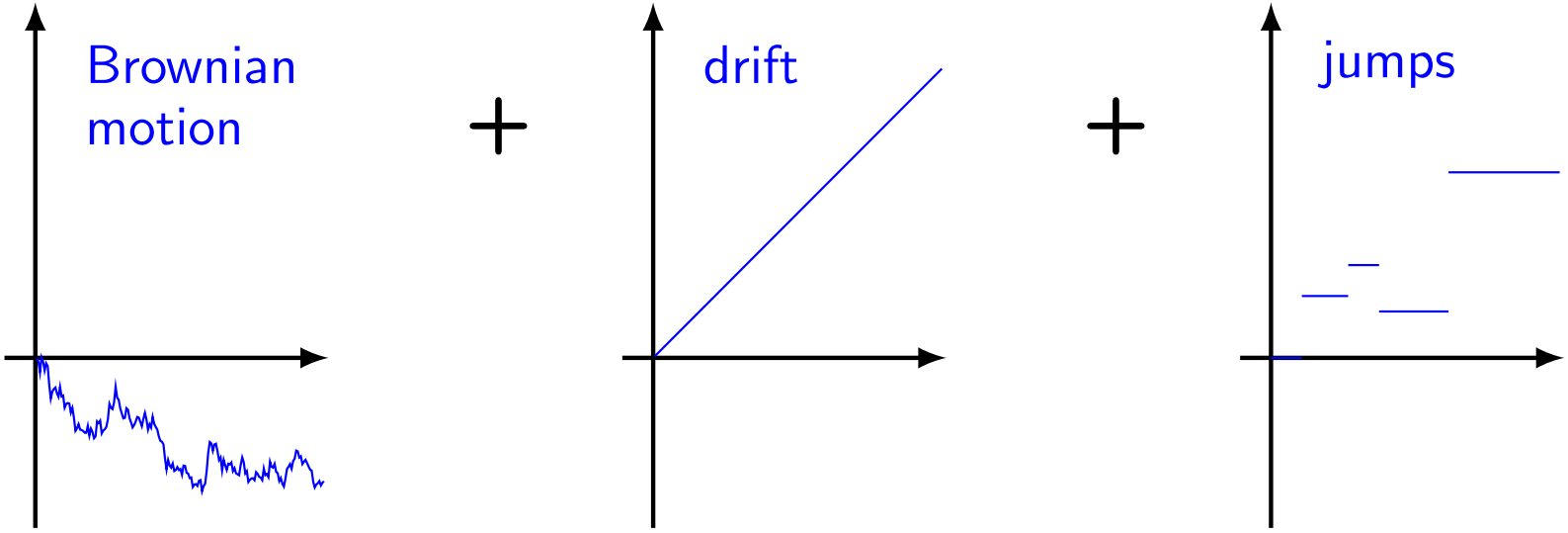
Modeling with Lévy processes

Directly modeling of the log-returns

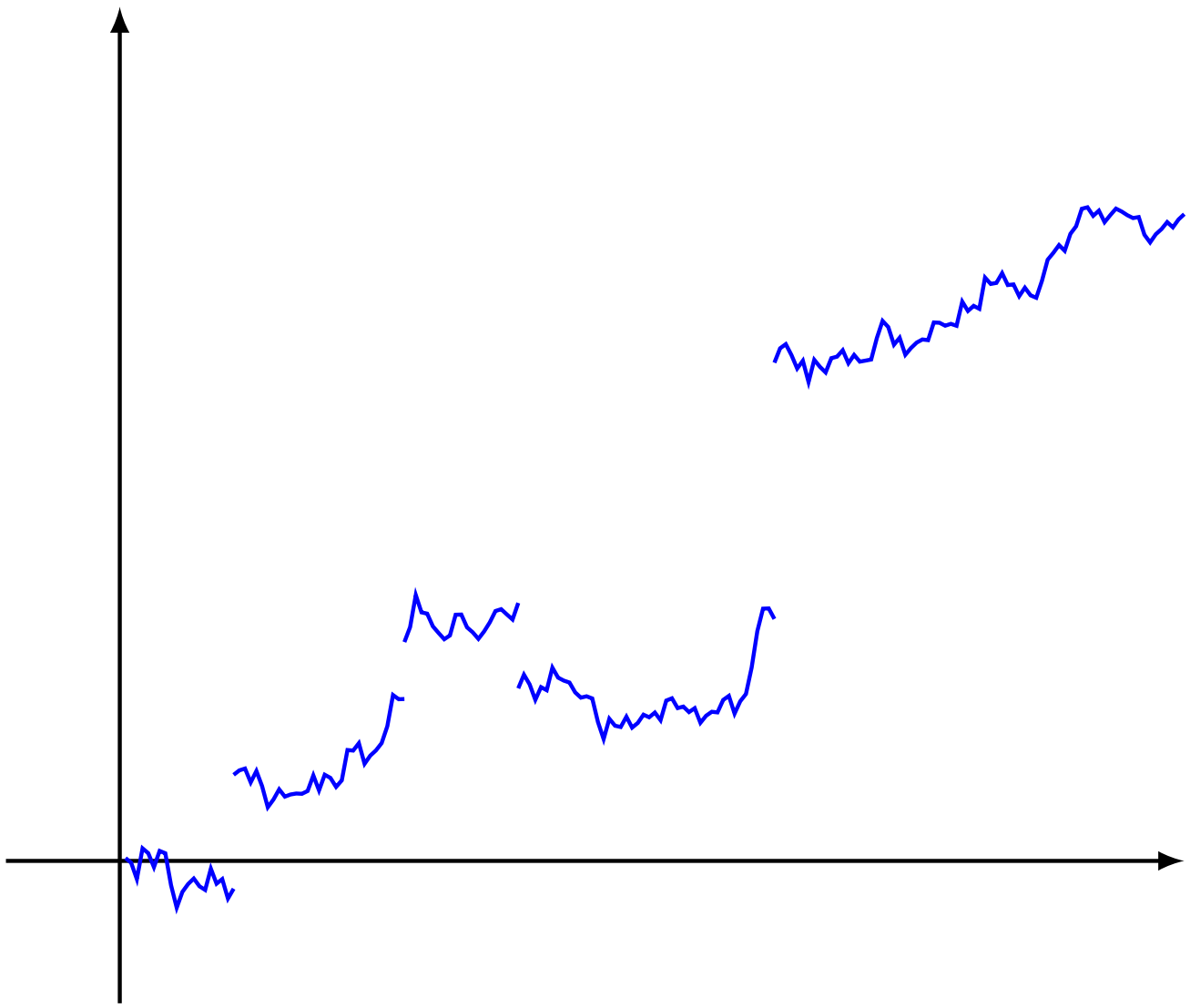
$$S_t = S_0 \, \mathrm{e}^{X_t}, \quad \text{for every } t \geq 0.$$

Consider *log returns* $(X_{t_n} - X_{t_{n-1}})$ iid but not any longer normally distributed as Black & Scholes assumed.

Lévy processes:



Path of a Lévy process



Lévy processes

Definition

A stochastic process $(X_t)_{t\geq 0}$ *with* $X_0 = 0$ *is called a* **Lévy process**, *if*

(L1) X *has independent increments, i.e.* $X_t - X_s$ *is independent of* $\mathcal{F}_s$ *for every* $t \geq s$,

(L2) X *has stationary increments, i.e.* $P_{X_{t+h}-X_t} = P_{X_h}$ *for every* $h > 0$, *and*

(L3) X *is stochastically continuous, i.e.*

$$\lim_{s\to t} P(|X_t - X_s| > \epsilon) = 0$$

for every $t \geq 0$ *and* $\epsilon > 0$.

Lévy model

Model: $S_t = S_0\, \mathrm{e}^{L_t}$, $L_t = \sigma B_t + \mu t + L_t^d$

- B Brownian motion, L^d pure jump process

Lévy-Khintchine formula $E\, \mathrm{e}^{iuL_t} = \mathrm{e}^{-tA(-u)}$ with **symbol**

$$A(u) = \frac{\sigma^2}{2} u^2 + ibu - \int \left(\mathrm{e}^{-iuy} - 1 + iuy \right) F(\mathrm{d}y)$$

- Approach to Lévy models: via **Fourier transform**

PIDE for option pricing

$$\Pi_t^g = E[g(L_T)\,|\mathcal{F}_t] = \underbrace{E[g(L_{T-t}+x)]}_{=:u(t,x)}\Big|_{x=L_t} =: \Gamma_{T-t}\, g(x)\Big|_{x=L_t}$$

$$\partial_t u(t,x) = -\lim_{h\downarrow 0}\frac{\Gamma_h - 1}{h}\, u(t,x) = -\,\mathcal{G}\; u(t,x)$$

by the definition of the inf. generator $\mathcal{G}$ of L i.e.

$$\partial_t u + \mathcal{G}\, u = 0\,, \qquad u(T,x) = g(x)$$

PIDE for option pricing

$$\Pi_t^g = E[g(L_{T-t} + x)]\,\big|_{x=L_t} =: u(t, L_t)$$

$$\partial_t u + \mathcal{G}\, u = 0\,, \qquad u(T, x) = g(x)$$

with infinitesimal generator $\mathcal{G} = \frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu\frac{\mathrm{d}}{\mathrm{d}x} + \mathcal{G}^{\text{jump}}$

$$\mathcal{G}^{\text{jump}}\varphi(x) = \int \Big(\varphi(x+y) - \varphi(x) - h(y)\varphi'(x)\Big) F(\mathrm{d}y)$$

PDE methods for pricing

- Pricing path-dependent options:
  E.g. American and barrier options.
- Option pricing beyond Lévy models.

Arsenal of Galerkin methods:

- Standard Finite Element (Achdou, Pironneau, Schwab, etc.)
- Wavelet-Galerkin (Schwab)
- Reduced Basis (Cont, Pironneau).

Outline

We pose the following questions:

(1) *Under which conditions on the symbol is the Kolmogorov equation parabolic? With tractable solution space?*

Sobolev-Slobodeckii space, Sobolev index of a Lévy process

(2) *How to adapt the concept for realistic option payoffs?*

Exponentially weighted spaces

(3) *Does the variational solution yield the option price?*

Feynman-Kac formula for variational solutions

Galerkin Method

Gelfand triplet $V \hookrightarrow H \hookrightarrow V^*$ and $\mathcal{A} : V \to V^*$ linear with bilinear form

$$a(u, v) := (\mathcal{A}\, u)(v) = \langle \mathcal{A}\, u, v \rangle_{V^* \times V}.$$

Continuity $\quad a(u, w) \leq C_1 \|u\|_V \|w\|_V \qquad \forall u, w \in V,$

Gårding inequality $\quad a(u, u) \geq C_2 \|u\|_V^2 - C_3 \|u\|_H^2 \qquad \forall u \in V,$

with $C_1, C_2 > 0$ and $C_3 \geq 0$.

Galerkin Method

Variational (or weak) solutions of parabolic equations:

Theorem

For $f \in L^2\big(0, T; V^*\big)$ and $g \in H$,

$$\begin{aligned} \partial_t u + \mathcal{A}\, u &= f \\ u(0) &= g\,, \end{aligned}$$

has a unique solution $u \in W^1$, i.e. $u \in L^2\big(0, T; V\big)$ with $\dot{u} \in L^2\big(0, T; V^*\big)$.

Sobolev index of a Lévy process

Fourier transform of $\mathcal{A}\,u$:

$\mathcal{A}$ Pseudo Differential Operator $\mathcal{F}\big(\mathcal{A}\,u\big) = A\,\mathcal{F}(u)$

Symbol $A(\xi) = \dfrac{\sigma^2}{2}\xi^2 + ib\xi - \displaystyle\int \big(\mathrm{e}^{-i\xi y} - 1 - i\xi h(y)\big) F(\mathrm{d}y)$

Sobolev index of a Lévy process

Bilinear form via Symbol for $u, v \in C_0^\infty$

$$a(u,v) = \int (\mathcal{A}\, u) v = \frac{1}{2\pi} \int A\, \hat{u}\, \overline{\hat{v}} \quad \text{(Parseval)}$$

Sobolev-Slobodeckii space $H^s := \left\{ u \in L^2 \mid \|u\|_{H^s} < \infty \right\}$ with norm

$$\|u\|_{H^s}^2 = \int \left|\hat{u}(\xi)\right|^2 \left(1 + |\xi|^s\right)^2 \mathrm{d}\xi$$

Sobolev index of a Lévy process

Definition

Let A be a symbol (and L be a Lévy process with symbol A). We call $\alpha \in (0,2]$ the **Sobolev index of the symbol** *A (resp. of the process L), if*

Continuity condition $\quad \big|A(\xi)\big| \leq C_1\big(1+|\xi|\big)^{\alpha}$

Gårding condition $\quad \Re\big(A(\xi)\big) \geq C_2\big(1+|\xi|\big)^{\alpha} - C_3\big(1+|\xi|\big)^{\beta}$

for $\xi \in \mathbb{R}$ with $C_1, C_2 > 0$ and $C_3 \geq 0$, $0 \leq \beta < \alpha$.

Sobolev index of a Lévy process

$$\begin{aligned} \dot{u} + \mathcal{A}\, u &= f \\ u(0) &= g \end{aligned} \tag{1}$$

with $f \in L^2\big(0, T; H^{-s}(\mathbb{R}^d)\big)$ and initial condition $g \in L^2(\mathbb{R}^d)$.

Theorem

Let $\mathcal{A}$ be a PDO with symbol A and bilinear form a.
Then (i) $\Longrightarrow$ (ii). If additionally A is continuous, (i) $\Longleftrightarrow$ (ii).

(i) **The symbol A has a Sobolev index $\alpha \in (0, 2]$.**

(ii) **The evolution problem (1) is parabolic** *w.r.t. Gelfand triplet $\big(H^{\alpha/2}(\mathbb{R}^d), L^2(\mathbb{R}^d), H^{-\alpha/2}(\mathbb{R}^d)\big)$ and has a unique variational solution $u \in W^1\big(0, T; H^{\alpha/2}(\mathbb{R}^d), L^2(\mathbb{R}^d)\big)$.*

Sobolev index measures smoothing

Proposition

If the Lévy process has Sobolev index α, then for every $t > 0$, P^{L_t} has a smooth and bounded Lebesgue density.

Let $\widetilde{H}^{\alpha/2}(D) := \{u \in H^{\alpha/2}(\mathbb{R}^d) | u|_{D^c} = 0\}$.
Set $a := a|_{\widetilde{H}^{\alpha/2}(D) \times \widetilde{H}^{\alpha/2}(D)}$.

Corollary

Let $\mathcal{A}$ be a PDO with symbol A and Sobolev index α. Let $D \subset \mathbb{R}^d$ be open with the segment property. Then

$$\begin{aligned} \partial_t u + \mathcal{A}\, u =& f \\ u(0) =& g\,, \end{aligned}$$

for $f \in L^2\big(0, T; \widetilde{H}^{-\alpha/2}(D)\big)$ and initial condition $g \in L^2(D)$ has a unique weak solution u in the space $W^1\big(0, T; \widetilde{H}^{\alpha/2}(D), L^2(D)\big)$.

Blumenthal-Getoor index

For a Lévy process with characteristics (b, c, F),

$$\beta := \inf\left\{\gamma > 0 \,\middle|\, \int_{[-1,1]} |x|^\gamma F(\mathrm{d}x) < \infty\right\}$$

is called the *Blumenthal-Getoor index* of the process.

Proposition

Let L be a Lévy process with characteristics $(b, 0, F)$ and Blumenthal-Getoor index β.

(a) *If $\beta < 1$ or $\beta = 1$ and $\int_{-1}^{1} |x| F(\mathrm{d}x) < \infty$, then P-a.e. path is of bounded variation on $(0, t]$ $\forall t > 0$.*

(b) *If $\beta > 1$ or $\beta = 1$ and $\int_{-1}^{1} |x| F(\mathrm{d}x) = \infty$, then P-a.e. path is of unbounded variation on $(0, t]$ $\forall t > 0$.*

Sobolev index and Blumenthal-Getoor index

Theorem

For an $\mathbb{R}$-valued Lévy process with characteristics $(b, 0, F)$ and Sobolev index $0 < \alpha < 2$, α equals the Blumenthal-Getoor index.

Question: Given L has Blumenthal-Getoor index β find sufficient conditions such that β is the Sobolev index.

Sobolev index and Blumenthal-Getoor index

Example

Let L be a Lévy process with characteristics $(0, 0, F)$ with Lévy density $F(\mathrm{d}x) := -|x|^{-1-\alpha} \log(|x|) \mathbb{1}_{[-1,1]}(x)\, \mathrm{d}x$. Then

$$\int_{-1}^{1} |x|^{\alpha-\epsilon} F(\mathrm{d}x) = \infty, \quad \int_{-1}^{1} |x|^{\alpha+\epsilon} F(\mathrm{d}x) < \infty.$$

I.e. α is the Blumenthal-Getoor index of L. Moreover,

$$A(u) \geq |u|^{\alpha} \log(|u|) \int_{-|u|}^{|u|} \big(1 - \cos(y)\big) |y|^{-1-\alpha}\, \mathrm{d}y,$$

hence the continuity condition is **not** satisfied with index α.

Sobolev index and Blumenthal-Getoor index

Proposition

Let L be an $\mathbb{R}$-valued Lévy process with characteristics $(b, 0, F)$. Let $0 < \beta < 2$.

(i) *If $\int_{[-1,1]} |x|^\beta F(\mathrm{d}x) < \infty$ and $|\Im(A(u))| \leq C(1 + |u|^\beta)$, then the continuity condition is satisfied with index $\alpha := \beta$.*

(ii) *If $\liminf_{r \downarrow 0} r^{\alpha-2} \int_{[-r,r]} |x|^2 F(\mathrm{d}x) > 0$, then the Gårding condition is satisfied with index $\alpha := \beta$.*

Sobolev index: Examples

Example

(i) Every pure jump Lévy process which has a Sobolev index has infinite jump activity, i.e. $F(\mathbb{R}) = \infty$.

(ii) For every Lévy process with Sobolev index $\alpha > 1$, we have $\int_{[-1,1]} |x| F(\mathrm{d}x) = \infty$.

(iii) Compound Poisson processes do not have a Sobolev index.

(iv) Variance gamma processes do not have a Sobolev index.

Sobolev index: Examples

Example (Multivariate α-semi-stable Lévy process)

Let L be a nondegenerate $\mathbb{R}^d$-valued α-semi-stable Lévy process.

a) If $0 < \alpha < 1$, then L has Sobolev index α if and only if L is strictly α-semi-stable.

b) If $1 < \alpha \leq 2$, then L has Sobolev index α.

Let $L \neq 0$ be an $\mathbb{R}$- valued 1-stable Lévy process.

c) L has Sobolev index 1 if and only if L is strictly 1-stable.

Sobolev index: Examples

Distribution	Sobolev index α
Brownian motion (+ drift) (+jumps)	2
GH, NIG	1
CGMY, no drift	Y
CGMY $Y \geq 1$	Y
Student-t	1
Cauchy	1
Variance Gamma	–

Sobolev index of a Lévy process

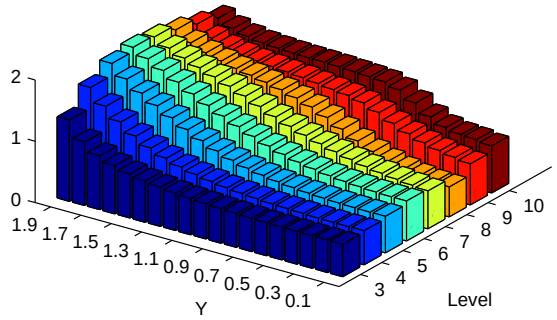


Figure: Values of experimental order of convergence, $EOC(l)$ for different values of Y and different levels l with fixed level $L = 14$.

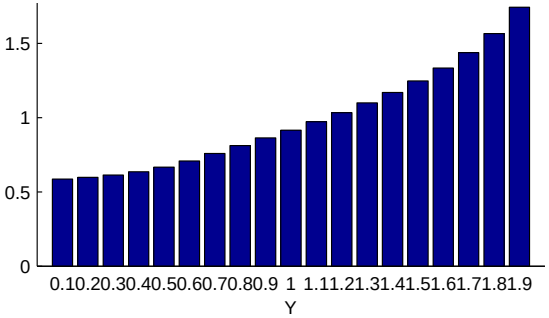


Figure: Average $\frac{1}{9}\sum_{l=3}^{l=11} EOC(l)$ for different values of Y.

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- Glau, *Classification of Lévy processes with parabolic Kolmogorov backward equations*, submitted

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(2) *How to adapt the concept for realistic option payoffs?*

Exponentially weighted spaces

(3) *Does the variational solution yield the option price?*

Feynman-Kac formula for variational solutions

PIDE and option pricing

L^2-based approach: Initial condition $g \in L^2$

$$\text{but } (S_T - K)^+ = (\mathrm{e}^{L_T} - K)^+ = g(L_T) \quad \Longrightarrow \quad g \notin L^2$$

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exponential weighting

$$g \in L^2_\eta \quad :\Leftrightarrow \quad g_\eta = g\,\mathrm{e}^{\langle \eta, \cdot \rangle} \in L^2$$

weighted Sobolev spaces H^s_η

$$\|u\|_{H^s_\eta} = \|u_\eta\|_{H^s} = \int \left|\mathcal{F}(u_\eta)(\xi)\right|^2 \left(1 + |\xi|\right)^{2s} \mathrm{d}\xi$$

PIDE and option pricing

exponentially weighted spaces

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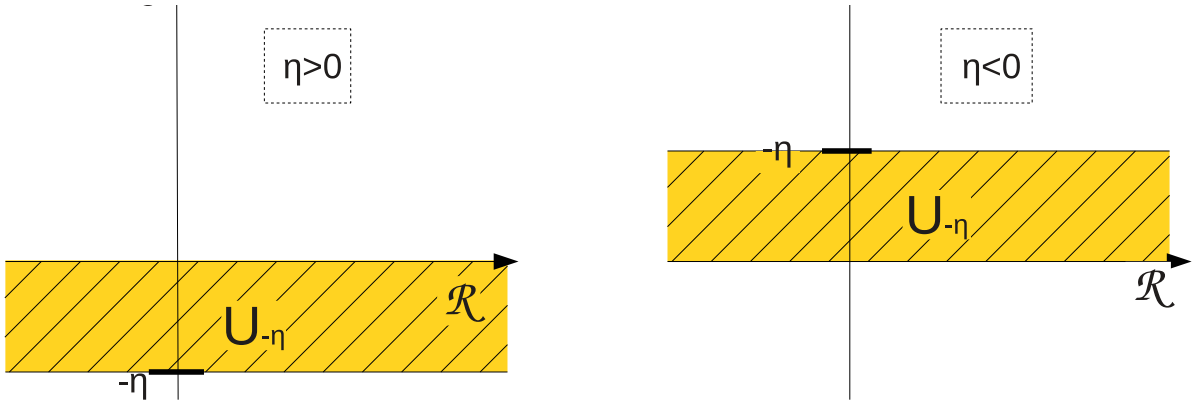
shifting the symbol in the complex plane



PIDE and option pricing

Let $\eta \in \mathbb{R}^d$ with

(A1) Exponential moments $\int_{|x|>1} |\, \mathrm{e}^{\langle z,x\rangle} \,| F(\mathrm{d}x) < \infty, \quad \forall z \in U_{-\eta}$.

(A2) **Continuity** $\big|A(z)\big| \leq C_1\big(1+|z|\big)^{\alpha}$.

(A3) **Gårding** $\Re\big(A(z)\big) \geq C_2\big(1+|z|\big)^{\alpha} - C_3\big(1+|z|\big)^{\beta}$.

For all $z \in U_{-\eta}$
with constants $C_1,\ C_2 > 0,\ C_3 \geq 0$
and $0 \leq \beta < \alpha$.

PIDE for European options

Theorem

Let (A1)–(A3) with $\alpha \geq 1$ and $\eta > 0$. Then $\exists!\, u \in W^1$ which is the weak solution of

$$\partial_t u + \mathcal{A}\, u = 0 \quad \text{in } \left(H_\eta^{\alpha/2}\right)^*$$
$$u(0) = g.$$

Outline

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- Glau, *Classification of Lévy processes with parabolic Kolmogorov backward equations*, submitted

(2) *How to adapt the concept for realistic option payoffs?*

- Eberlein and Glau, *Variational solutions of the pricing PIDEs for European options in Lévy models*, to appear in Applied Mathematical Finance

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(3) *Does the variational solution yield the option price?*

Feynman-Kac formula for variational solutions

Feynman-Kac formula for processes with killing rate

- Now we generalize from Lévy processes to time-inhomogeneous processes and PIIAC.

Consider the generalized Kolmogorov backward equation

$$\partial_t u + \mathcal{A}_{T-t} u + \kappa_{T-t} u = f, \quad u(0) = g\,,$$

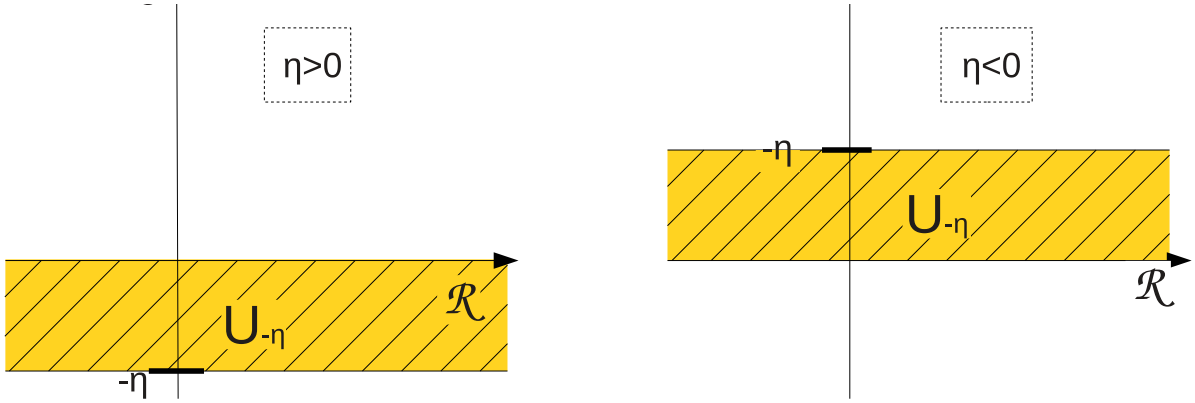
with a bounded **killing rate** κ.

Question: Under which conditions $\exists!$ weak solution with stochastic representation

$$u(T-t, L_t) = E\Big(g(L_T)\,\mathrm{e}^{-\int_t^T \kappa_h(L_{h-})\,\mathrm{d}h} + \int_t^T f(T-s, L_s)\,\mathrm{e}^{-\int_t^s \kappa_h(L_{h-})\,\mathrm{d}h}\,\mathrm{d}s \,\Big|\, \mathcal{F}_t\Big)$$

Assumptions

Let L be an $\mathbb{R}^d$-valued PIIAC and let $A : (t, z) \to A_t(z)$, $A : [0, T] \times U_{-\eta} \to \mathbb{C}$ and fix some $\alpha > 0$ and some $\eta \in \mathbb{R}^d$.



- d-dimensional: $U_{-\eta} = U_{-\eta^1} \times \cdots \times U_{-\eta^d}$
- Define R_η as cartesian product of imaginary parts of $U_{-\eta}$.

Assumptions

(A1) For the Lévy measure F of L we have for all $\eta' \in R_\eta$.

$$\int_0^T \int_{|x|>1} \mathrm{e}^{2|\langle \eta', x\rangle|} F_s(\mathrm{d}x)\,\mathrm{d}s < \infty. \qquad \textbf{(exponential moments)}$$

(A2) There exists a constant $C_1 > 0$ with

$$\left|A_t(\xi + i\eta')\right| \leq C_1\left(1 + |\xi|\right)^\alpha \qquad \textbf{(continuity condition)}$$

for all $\xi \in \mathbb{R}^d$, all $\eta' \in R_\eta$ and for all $t \in [0, T]$.

(A3) There exist constants $C_2 > 0$ and $C_3 \geq 0$, such that for a $0 \leq \beta < \alpha$

$$\Re\big(A_t(\xi + i\eta')\big) \geq C_2\left(1 + |\xi|\right)^\alpha - C_3\left(1 + |\xi|\right)^\beta \quad \textbf{(Gårding condition)}$$

for all $\xi \in \mathbb{R}^d$, all $\eta' \in R_\eta$ and for all $t \in [0, T]$.

(A4) The mapping $t \mapsto A_t(\xi - i\eta)$ is continuous for every fixed $\xi \in \mathbb{R}^d$.

(A5) The mapping $t \mapsto \Re\big(A_t(\xi - i\eta)\big)$ is continuously differentiable for every fixed $\xi \in \mathbb{R}^d$ with

$$\left|\partial_t \Re(A_t(\xi - i\eta))\right| \leq C_4\left(1 + |\xi|\right)^\alpha \qquad \text{for all } \xi \in \mathbb{R}^d.$$

Feynman-Kac Representation

Theorem (Feynman-Kac Representation)

Let L be an $\mathbb{R}^d$-valued PIIAC satisfying assumptions (A1)–(A5). Let $f \in L^2\big(0, T; H^1_\eta(\mathbb{R}^d)\big)$, $\kappa : [0, T] \times \mathbb{R}^d \to [0, \infty)$ be bounded such that $t \mapsto \kappa_t(x)$ is continuous for a.e. $x \in \mathbb{R}^d$, and $g \in L^2_\eta(\mathbb{R}^d)$.

Then the unique weak solution $u \in W^1$ of

$$\partial_t u + \mathcal{A}_{T-t} u + \kappa_{T-t} u = f, \quad u(0) = g\,,$$

has for every $t \in (0, T]$ the stochastic representation

$$u(T-t, L_t) = E\Big(g(L_T)\,\mathrm{e}^{-\int_t^T \kappa_h(L_{h-})\,\mathrm{d}h} + \int_t^T f(T-s, L_s)\,\mathrm{e}^{-\int_t^s \kappa_h(L_{h-})\,\mathrm{d}h}\,\mathrm{d}s \,\Big|\, \mathcal{F}_t\Big) \quad a.s.$$

Proof of Feynman-Kac Representation

Sketch of proof:

- Continuity and Gårding condition for $\mathcal{A} + \kappa$ are trivial.
- A regularity assertion implies

$$\dot{u} + \mathcal{A}\, u + \kappa u = f \quad \text{a. e.}$$

For $w(t,x) := u(T-t,x)$ we have

$$\int [\dot{w} + \mathcal{A}_s w + \kappa w](s, L_s)\, \mathrm{d}s = \int -f(T-s, L_s)\, \mathrm{d}s$$

Proof of Feynman-Kac Representation

- Itô requires higher regularity, so approximate w by a smooth sequence. Then:

$$
\begin{aligned}
&w^n(s, L_s) \\
&= E\Big(w^n(t, L_t)\, \mathrm{e}^{-\int_s^t \kappa_h(L_{h-})\,\mathrm{d}h}\Big) \\
&\quad + E\Big(\int_s^t \Big(\dot{w}^n - \mathcal{A}_h w^n - \kappa w^n\Big)(h, L_{h-})\, \mathrm{e}^{-\int_s^h \kappa_\lambda(L_{\lambda-})\,\mathrm{d}\lambda}\,\mathrm{d}h \Big| \mathcal{F}_s\Big).
\end{aligned}
$$

Taking the limit $n \to \infty$ yields

$$
\begin{aligned}
&w(s, L_s) \\
&= E\Big(w(t, L_t)\, \mathrm{e}^{-\int_s^t \kappa_h(L_{h-})\,\mathrm{d}h} + \int_s^t f(T-h, L_{h-})\, \mathrm{e}^{-\int_s^h \kappa_\lambda(L_{\lambda-})\,\mathrm{d}\lambda}\,\mathrm{d}h \Big| \mathcal{F}_s\Big).
\end{aligned}
$$

Hölder continuity

Corollary

We have for a.e. $x \in \mathbb{R}^d$

$$u(T,x) = E_x\Big(g(L_T)\,\mathrm{e}^{-\int_0^T \kappa_h(L_{h-})\,\mathrm{d}h} + \int_0^T f(T-s, L_s)\,\mathrm{e}^{-\int_0^s \kappa_h(L_{h-})\,\mathrm{d}h}\,\mathrm{d}s\Big).$$

- *For* $\alpha \in (1,2]$ *and* $d = 1$, $x \mapsto u(t,x)$ *is* λ*-Hölder continuous with* $\lambda = \frac{\alpha-1}{2}$ *for any fixed* $t \in (0,T)$.
- *In particular,* $x \mapsto u(t,x)$ *is continuous and the expression for* $u(T,x)$ *holds for every* $x \in \mathbb{R}$.

$\varphi : \mathbb{R}^d \to \mathbb{R}$ is λ-Hölder continuous iff

$$\sup_{x,y\in\mathbb{R}, x\neq y} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\lambda} < \infty.$$

Applications

- For killing rates of indicator type, $\kappa := \lambda \mathbb{1}_D$ for $D \subset \mathbb{R}^d$ and $\lambda > 0$, initial condition $g \equiv 1$, $f \equiv 0$ and $L_0 = x$, we get

$$u(T, x) = E_x\big(\mathrm{e}^{-\lambda \int_0^T \mathbb{1}_D(L_{h-})\,\mathrm{d}h} \big),$$

  which is the **Laplace transform** at λ of the **occupation time** $\int_0^T \mathbb{1}_D(L_{h-})\,\mathrm{d}h$ of L spent until time T in the domain D.

- **Ruin theory**: Replacing concept of ruin by bankruptcy, see
  - Albrecher and Lautscham (2013) and
  - Albrecher, Gerber and Shiu (2011).
- Employee stock options,
- relativistic quantum mechanics, see
  Carmona, Masters and Simon, *Relativistic Schrödinger Operators: Asymptotic Behavior of the Eigenfunctions*, Journal of Functional Analysis, 1990.

(1) *Under which conditions on the symbol is the Kolmogorov equation parabolic? With tractable solution space?*

- Glau, *Classification of Lévy processes with parabolic Kolmogorov backward equations*, submitted

(2) *How to adapt the concept for realistic option payoffs?*

- Eberlein and Glau, *Variational solutions of the pricing PIDEs for European options in Lévy models*, to appear in Applied Mathematical Finance

(3) *Does the variational solution yield the option price?*

- Glau, *Feynman-Kac formula for Lévy processes with killing rate and bankruptcy*, working paper
- Eberlein and Glau, *Feynman-Kac theorem for Lévy processes and weak solutions to boundary value problems*, work in prog.
- Glau, *Feynman-Kac-Darstellung zur Optionspreisbewertung in Lévy-Modellen*, PhD-Thesis

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Thank you for your attention!