Asymptotic analysis for optimal estimating functions for a class of stochastic volatility models with jumps

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Inference for BNS stochastic volatility models

- In the discretely observed Barndorff-Nielsen Shepard (BNS) setting we explore the joint distribution of spot prices X and the instantaneous variance V supposing that both X and V can be observed
- inference by the *martingale estimating function approach* leads to an **explicit** estimator
- *consistency* and *asymptotic normality* of the simple explicit estimator are shown

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The BNS model

Continuous time model

$$dX(t) = (\mu + \beta V(t-))dt + \sqrt{V(t-)}dW(t) + \rho dZ_\lambda(t), \quad X(0) = 0.$$

and

$$dV(t) = -\lambda V(t-)dt + dZ_\lambda(t), \quad V(0) = V_0,$$

where $\mu, \beta, \rho, \lambda \in \mathbb{R}$ with $\lambda > 0$. $Z = (Z_t)$ is the BDLP $Z_\lambda(t) = Z(\lambda t)$.
V_0 has a self-decomposable distribution corresponding to the BDLP s. t. the process V is strictly stationary and

$$E[V_0] = \zeta, \qquad Var[V_0] = \eta.$$

Assumption: $E[V_0^n] < \infty, \ \forall n \in \mathbb{N}$.
True for Γ-OU, IG-OU,.....

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Discretely observed continuous time model

Observing X and V on a discrete grid of points in time, $0 = t_0 < t_1 < \ldots < t_n$, we obtain:

$$V(t_i) = V(t_{i-1})e^{-\lambda(t_i-t_{i-1})} + \int_{t_{i-1}}^{t_i} e^{-\lambda(t_i-s)} dZ_\lambda(s)$$

and

$$
\begin{aligned}
X(t_i) - X(t_{i-1}) &= \mu(t_i - t_{i-1}) + \beta(Y(t_i) - Y(t_{i-1})) + \int_{t_{i-1}}^{t_i} \sqrt{V(s-)} dW(s) \\
&+ \rho(Z_\lambda(t_i) - Z_\lambda(t_{i-1})),
\end{aligned}
$$

where $\quad Y(t) = \int_0^t V(s-)ds.$

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and

$$X(t_i) - X(t_{i-1}) = \mu(t_i - t_{i-1}) + \beta(Y(t_i) - Y(t_{i-1})) + \int_{t_{i-1}}^{t_i} \sqrt{V(s-)} dW(s)$$
$$+ \rho(Z_\lambda(t_i) - Z_\lambda(t_{i-1})),$$

where $Y(t) = \int_0^t V(s-)ds$.

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Introducing

$$X_i = X(t_i) - X(t_{i-1}), \quad Y_i = Y(t_i) - Y(t_{i-1}), \quad Z_i = Z_\lambda(t_i) - Z_\lambda(t_{i-1}),$$

$$W_i = \frac{1}{\sqrt{Y_i}} \int_{t_{i-1}}^{t_i} \sqrt{V(s-)} dW(s) \overset{i.i.d}{\sim} N(0,1),$$

and an auxiliary quantity $U(t) = \int_0^t e^{-\lambda(t-s)} dZ_\lambda(s)$,

using $t_k = k$ and $V_i = V(t_i)$ we have

$$V_i = \gamma V_{i-1} + U_i$$

$$X_i = \mu + \beta Y_i + \sqrt{Y_i} W_i + \rho Z_i,$$

where $\gamma = e^{-\lambda}$, $\quad U_i = U(t_i) - U(t_{i-1})$, $\quad (U_i, Z_i)$ i.i.d

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Martingale estimating functions

- Suppose that $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are observations from a model with a *d*-dimensional parameter $\theta \in \Theta$.

- an estimator $\hat{\theta}_n$ is obtained solving the equation

$$G_n(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n; \theta) = 0, \tag{1}$$

where $G_n(\theta)$ is a d-dim estimating function of the parameter $\theta \in \mathbb{R}^d$.

- Among the class of unbiased or Fisher consistent estimating functions, we will analyze those estimating functions that are ***martingales***.

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Estimating equations

Let $\qquad \theta = (\lambda, \zeta, \eta, \beta, \rho, \mu)^\top, \qquad \mathbf{X} = (X, V).$

We consider the following 6 martingale estimating functions:

$$G_n^1(\theta) = \sum_{i=1}^n \left[V_i - E(V_i | V_{i-1}) \right], \quad G_n^2(\theta) = \sum_{i=1}^n \left[V_i V_{i-1} - V_{i-1} E(V_i | V_{i-1}) \right],$$

$$G_n^3(\theta) = \sum_{i=1}^n \left[V_i^2 - E(V_i^2 | V_{i-1}) \right], \quad G_n^4(\theta) = \sum_{i=1}^n \left[X_i - E(X_i | V_{i-1}) \right],$$

$$G_n^5(\theta) = \sum_{i=1}^n \left[X_i V_{i-1} - V_{i-1} E(X_i | V_{i-1}) \right], \quad G_n^6(\theta) = \sum_{i=1}^n \left[X_i V_i - E(X_i V_i | V_{i-1}) \right].$$

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The specifications for $G_n^1 - G_n^6$ belong to the more general class of martingale estimating functions of the form

$$G_n^j(\theta) = \sum_{i=1}^n \alpha^j(V_{i-1}; \theta)\big[X_i^{r_j} \cdot V_i^{s_j} - f^j(V_{i-1}; \theta)\big], \quad j = 1, \ldots, d,$$

where $\alpha^j(V_{i-1})$ is some $\mathcal{F}_{i-1}$-adapted random variable,

$$f^j(v, \theta) = E[X_1^{r_j} V_1^{s_j} | V_0 = v]$$

and we have

$$f^j(v; \theta) = \sum_{l=0}^{r_j+s_j} \phi_l^j(\theta) \cdot v^l.$$

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Remark

- for the simple explicit estimator $\alpha^j(v) = v$ or $\alpha^j(v) = 1$ which gives explicit $G_n(\theta)$, the explicit solution of $G_n(\theta) = 0$ and the explicit $Cov(\hat{\theta}_n)$.

- for **optimal** estimating functions $\alpha^j(v)$ are rational in v which gives explicit $G_n(\theta)$, explicit $Cov(\hat{\theta}_n)$ but must solve $G_n(\theta) = 0$ numerically

- in this setting the problem of finding the resulting estimator explicitly amounts to **solving** d **explicitly given equations**.

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The simple estimator $\hat{\theta}_n = (\lambda_n, \zeta_n, \eta_n, \beta_n, \rho_n, \mu_n)$ is given by

$$\lambda_n = \log\left[\frac{M_{V_0^2}^n - (M_{V_0}^n)^2}{M_{V_1 V_0}^n - M_{V_1}^n M_{V_0}^n}\right], \quad \zeta_n = \frac{1}{1 - e^{-\lambda_n}}\left[M_{V_1}^n - e^{-\lambda_n} M_{V_0}^n\right],$$

$$\eta_n = \frac{1}{1 - e^{-2\lambda_n}}\left[\left(M_{V_1^2}^n - (M_{V_1}^n)^2\right) - e^{-2\lambda_n}\left(M_{V_0^2}^n - (M_{V_0}^n)^2\right)\right],$$

$$\beta_n = \frac{M_{X_1 V_0}^n - M_{X_1}^n M_{V_0}^n}{M_{Y_1 V_0}^n - M_{Y_1}^n M_{V_0}^n}, \quad \mu_n = M_{X_1}^n - \beta_n M_{Y_1}^n - \rho_n M_{Z_1}^n,$$

$$\rho_n = \frac{1}{M_{Z_1 V_1}^n - M_{Z_1}^n M_{V_1}^n}\left[M_{X_1 V_1}^n - M_{X_1}^n M_{V_1}^n - \beta_n\left(M_{Y_1 V_1}^n - M_{Y_1}^n M_{V_1}^n\right)\right],$$

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$$M_{V_0}^n = \frac{1}{n} \sum_{i=1}^{n} V_{i-1}, \qquad M_{V_0^2}^n = \frac{1}{n} \sum_{i=1}^{n} V_{i-1}^2,$$

$$M_{X_1 V_0}^n = \frac{1}{n} \sum_{i=1}^{n} X_i V_{i-1}, \qquad M_{V_1}^n = \frac{1}{n} \sum_{i=1}^{n} V_i, \qquad M_{V_1 V_0}^n = \frac{1}{n} \sum_{i=1}^{n} V_i V_{i-1},$$

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$$M_{Z_1}^n = \zeta_n \lambda_n, \qquad M_{Z_1 V_1}^n = e^{-\lambda_n} \lambda_n \zeta_n M_{V_0}^n + (1 - e^{-\lambda_n})(2\eta_n + \lambda_n \zeta_n^2),$$

$$M_{Y_1}^n = \zeta_n - \frac{1}{\lambda_n}(M_{V_1}^n - M_{V_0}^n), \qquad M_{Y_1 V_0}^n = \zeta_n M_{V_0}^n - \frac{1}{\lambda_n}(M_{V_1 V_0}^n - M_{V_0^2}^n),$$

$$M_{Y_1 V_1}^n = \frac{1}{\lambda_n}(M_{Z_1 V_1}^n - M_{V_1^2}^n + M_{V_1 V_0}^n).$$

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Consistency and asymptotic normality

Theorem

*The estimator $\hat{\theta}_n = (\lambda_n, \zeta_n, \eta_n, \beta_n, \rho_n, \mu_n)$ is **consistent**,*

$$\hat{\theta}_n \xrightarrow{a.s.} \theta_0 \qquad as\ n \to \infty.$$

Theorem

*The estimator $\hat{\theta}_n = (\lambda_n, \zeta_n, \eta_n, \beta_n, \rho_n, \mu_n)$ is **asymptotically normal**,*

$$\sqrt{n}[\hat{\theta}_n - \theta_0] \xrightarrow{\mathcal{D}} N(0, T), \qquad as\ n \to \infty$$

*where T can be calculated **explicitly**.*

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Remarks

- previous analysis relied on the explicit solutions of the estimating function

- for studying optimal (quadratic) estimating functions, there is no explicit solution

- however, there is a general theory dealing with this problem

- it's instructive to review the simple estimator in this general framework

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Suppose:

$$G_n^j(\theta) = \sum_{i=1}^{n} \alpha^j(V_{i-1};\theta)\big[X_i^{r_j} V_i^{s_j} - f^j(V_{i-1};\theta)\big], \ j = 1,\ldots,d.$$

For $\theta \in \Theta$, let us define the $d \times d$ matrices

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Suppose:

$$G_n^j(\theta) = \sum_{i=1}^{n} \alpha^j(V_{i-1}; \theta)\big[X_i^{r_j} V_i^{s_j} - f^j(V_{i-1}; \theta)\big], \; j = 1, \ldots, d.$$

For $\theta \in \Theta$, let us define the $d \times d$ matrices

- $J_n(\theta) = \partial_{\theta^\tau} G_n(\theta)$
- $Q_n^{(j)}(\theta) = \partial_\theta^2 G_n^j(\theta).$

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For $\theta \in \Theta$, let us define the $d \times d$ matrices

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Condition 2.6. of Sørensen (1999)

(i) the mapping $\theta \longmapsto G_n(\theta)$ is twice continuously differentiable;

(ii) there exists a $\bar{\theta} \in \text{int } \Theta$ and an invertible non-random $d \times d$ matrix $A(\bar{\theta})$ such that

$$\sup_{\theta^{(i)} \in M_n^\alpha(\bar{\theta})} \| \frac{1}{n} J_n(\theta^{(1)}, \dots, \theta^{(d)}) - A(\bar{\theta}) \| \xrightarrow{P} 0$$

for all $\alpha > 0$;

(iii) there exist d non-random matrices $B^i(\bar{\theta})$, $i = 1, \dots, d$ such that

$$\sup_{\theta^{(i)} \in M_n^\alpha(\bar{\theta})} \| \frac{1}{n} Q_n^{(i)}(\theta^{(1)}, \dots, \theta^{(d)}) - B^i(\bar{\theta}) \| \xrightarrow{P} 0$$

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Condition 2.6. of Sørensen (1999)

(i) the mapping $\theta \longmapsto G_n(\theta)$ is twice continuously differentiable;

(ii) there exists a $\bar{\theta} \in \text{int } \Theta$ and an invertible non-random $d \times d$ matrix $A(\bar{\theta})$ such that

$$\sup_{\theta^{(i)} \in M_n^\alpha(\bar{\theta})} \| \frac{1}{n} J_n(\theta^{(1)}, \ldots, \theta^{(d)}) - A(\bar{\theta})\| \xrightarrow{P} 0$$

for all $\alpha > 0$;

(iii) there exist d non-random matrices $B^i(\bar{\theta}),\ i = 1, \ldots, d$ such that

$$\sup_{\theta^{(i)} \in M_n^\alpha(\bar{\theta})} \| \frac{1}{n} Q_n^{(i)}(\theta^{(1)}, \ldots, \theta^{(d)}) - B^i(\bar{\theta})\| \xrightarrow{P} 0$$

for all $\alpha > 0$;

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(iv) $\left\{ \dfrac{G_n(\bar{\theta})}{\sqrt{n}} : n \in \mathbb{N} \right\}$ is stochastically bounded;

(v) $\sup\limits_{\theta \in M_n^\alpha(\bar{\theta})} \left\| \dfrac{G_n(\theta)}{n} \right\| \xrightarrow{P} 0$ for all $\alpha > 0$.

Theorem

For every n an estimator $\hat{\theta}_n$ exists that solves the estimating equation $G_n(\hat{\theta}_n) = 0$ with a probability tending to one as $n \to \infty$ and $\hat{\theta}_n \xrightarrow{P} \bar{\theta}$ as $n \to \infty$. Moreover, if

$$\frac{1}{\sqrt{n}} G_n(\bar{\theta}) \xrightarrow{D} N(0, \Upsilon)$$

as $n \to \infty$, then

$$\sqrt{n}(\hat{\theta}_n - \bar{\theta}) \xrightarrow{D} N(0, A(\bar{\theta})^{-1} \Upsilon (A(\bar{\theta})^{-1})^\top)$$

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O_F and O_A optimality

Outline

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   - Discretely observed continuous time model

2 The simple explicit estimator
   - Estimating functions
   - The explicit solution
   - The general framework

3 Optimal estimating functions
   - O_F and O_A optimality

4 Possible future work

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O_F and O_A **optimality**

O_F-optimality

Definition

$G_T^* \in \mathcal{H}$ is an O_F-optimal estimating function within $\mathcal{H}$ if

$$\mathcal{E}(G_T^*) - \mathcal{E}(G_T)$$

is nonnegative definite for all $G_T \in \mathcal{H}$, $\theta \in \Theta$ and P_θ, where

$$\mathcal{E}(G_T) = E(\dot{G}_T)'(EG_T G_T')^{-1}(E\dot{G}_T).$$

- Not a very intuitive definition, but the idea is that if the score exists, an optimal estimating function within $\mathcal{H}$ is one with **minimum dispersion distance from the score**.

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O_F and O_A **optimality**

O_A-optimality

Definition

$G_T^* \in \mathcal{H}$ is an O_A-optimal estimating function within $\mathcal{H}$ if

$$\bar{G}_T^*(\theta)' \langle G^*(\theta) \rangle_T^{-1} \bar{G}_T^*(\theta) - \bar{G}_T'\langle G(\theta) \rangle_T^{-1} \bar{G}_T(\theta)$$

is almost surely nonnegative definite for all $G_T \in \mathcal{H}$, $\theta \in \Theta$, P_θ and $T > 0$, where $\dot{G}_t(\theta) - \bar{G}_T(\theta)$ is a martingale.

Relation between O_A and O_F optimality

O_A-optimality implies O_F-optimality in an important set of cases. The reverse implication does not ordinary hold.

Remark

The conditions needed for optimality are easier to verify when **G**s are **martingales**!

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is almost surely nonnegative definite for all $G_T \in \mathcal{H}$, $\theta \in \Theta$, P_θ and $T > 0$, where $\dot{G}_t(\theta) - \bar{G}_T(\theta)$ is a martingale.

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O_F and O_A optimality

Relation between O_A and O_F optimality

Theorem

Suppose that G_T^* is O_A-optimal within the convex class of martingale estimating functions $\mathcal{M}$. If $(\bar{G}_T^*)^{-1}\langle G^*\rangle_T$ is **non-random** for $T > 0$, then G_T^* is also O_F-optimal within $\mathcal{M}$.

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O_F **and** O_A **optimality**

Optimality in the class $\mathcal{M} \subset \mathcal{H}$ of quadratic estimating functions.

An estimating function in $\mathcal{M}$ is of the form

$$G_n(\theta) = \sum_{i=1}^{n} g(X_i, V_i, V_{i-1}, \theta),$$

where $g(x_1, v_1, v_0, \theta) = \sum_{j=1}^{N} \alpha_j(v_0, \theta) h_j(x_1, v_1, v_0, \theta), \ N = 5,$

$$
\begin{aligned}
h_1(x_1, v_1, v_0, \theta) &= v_1 - f_1(v_0, \theta), & f_1(v, \theta) &= E_\theta[V_1 | V_0 = v] \\
h_2(x_1, v_1, v_0, \theta) &= x_1 - f_2(v_0, \theta), & f_2(v, \theta) &= E_\theta[X_1 | V_0 = v] \\
h_3(x_1, v_1, v_0, \theta) &= v_1^2 - f_3(v_0, \theta), & f_3(v, \theta) &= E_\theta[V_1^2 | V_0 = v] \\
h_4(x_1, v_1, v_0, \theta) &= x_1^2 - f_4(v_0, \theta), & f_4(v, \theta) &= E_\theta[X_1^2 | V_0 = v] \\
h_5(x_1, v_1, v_0, \theta) &= x_1 v_1 - f_5(v_0, \theta), & f_5(v, \theta) &= E_\theta[X_1 V_1 | V_0 = v]
\end{aligned}
$$

and $\alpha_j(v, \theta)$ is a $p-$dimensional vector of measurable functions of v for each θ and $j = 1, \ldots, N$ such that $G_n(\theta)$ is **square integrable**.

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O_F and O_A optimality

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h_1(x_1, v_1, v_0, \theta) &= v_1 - f_1(v_0, \theta), & f_1(v, \theta) &= E_\theta[V_1 | V_0 = v] \\
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h_4(x_1, v_1, v_0, \theta) &= x_1^2 - f_4(v_0, \theta), & f_4(v, \theta) &= E_\theta[X_1^2 | V_0 = v] \\
h_5(x_1, v_1, v_0, \theta) &= x_1 v_1 - f_5(v_0, \theta), & f_5(v, \theta) &= E_\theta[X_1 V_1 | V_0 = v]
\end{aligned}
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and $\alpha_j(v, \theta)$ is a $p-$dimensional vector of measurable functions of v for each θ and $j = 1, \ldots, N$ such that $G_n(\theta)$ is **square integrable**.

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O_F and O_A optimality

Optimal estimating function in the sense of Heyde (1997) for the class $\mathcal{M}$.

Theorem

Define an estimating function by

$$G_n^*(\theta) = \sum_{i=1}^n g^*(X_i, V_i, V_{i-1}, \theta),$$

where

$$g^*(x_1, v_1, v_0, \theta) = A^*(v_0, \theta)h(x_1, v_1, v_0, \theta),$$

with

$$A^*(v; \theta) = B(v; \theta)C(v; \theta)^{-1} \qquad b_{ij}(v, \theta) = \partial_i f_j(v, \theta)$$

and

$$c_{ij}(v, \theta) = E_\theta[h_i(X_1, V_1, V_0, \theta)h_j(X_1, V_1, V_0, \theta)|V_0 = v]$$

Then G^* is O_A-optimal in $\mathcal{M}$ and G_n^* is O_F-optimal in $\mathcal{M}_n$ for all $n \in \mathbb{N}$.

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O_F and O_A optimality

Remark

- $b_{ij}(v)$ is a polynomial in v
- $c_{ij}(v)$ is a polynomial in v
- $\Longrightarrow$ $\alpha_{ij}^*(v)$ are explicit rational functions of v.

In fact

this allows us the application of the general framework given before and explicit calculation of the covariance matrix of θ_n.

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O_F and O_A optimality

Remark

- $b_{ij}(v)$ is a polynomial in v
- $c_{ij}(v)$ is a polynomial in v
  $\implies \qquad \alpha_{ij}^*(v) \qquad$ are explicit rational functions of v.

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this allows us the application of the general framework given before and explicit calculation of the covariance matrix of θ_n.

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O_F and O_A optimality

For consistency and AN of the optimal estimator, we use the general framework and results of Sørensen (1999).

Theorem

The estimator $\hat{\theta}_n$ obtained by solving the equation $G_n^*(\theta) = 0$ is asymptotically normal, namely

$$\sqrt{n}(\hat{\theta}_n - \bar{\theta}) \xrightarrow{\mathcal{D}} N\big(0, J(\bar{\theta})^{-1}\Phi(J(\bar{\theta})^{-1})^\top\big) \text{ as } n \to \infty$$

where

$$\Phi = \sum_{m=1}^{N}\sum_{z=1}^{N} E\big[\alpha_{im}^*(V_0, \bar{\theta})\alpha_{jz}^*(V_0, \bar{\theta})Cov(\Xi_1^m, \Xi_1^z | V_0)\big],$$

$$\Xi_k = \big(V_k, X_k, V_k^2, X_k^2, X_k V_k\big)^\top, \qquad k = 1, \dots, n,$$

$$J_n^{p,k}(\theta^{(1)}, \dots, \theta^{(d)}) = \frac{\partial G_n^p(\theta^{(p)})}{\partial \theta_k}, \qquad \theta^{(p)} \in \Theta,\ k = 1, \dots, d,\ p = 1, \dots, d$$

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Criticism to address

- Instantaneous variance is not an observable quantity in discrete time
- Various quantities are suggested as substitutes for the variance
- More efficient estimators than provided by the optimal quadratic estimating function can be obtained by incorporating further moments

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- Although very high moments are not reliable for empirical investigations, BNS-models allow also explicit computation of the characteristic function and thus of (un)conditional *trigonometric moments* $E[e^{i(\xi_k V_1 + \psi_k X_1)}]$ and $E[e^{i(\xi_k V_1 + \psi_k X_1)}|V_0]$ for arbitrary constants ξ_k and ψ_k, that could be used instead for constructing estimating functions.

- Comparison of our results to the related *generalized methods of moments*, which would require a precise specification of the weighting matrix

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