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`http://www.math.tu-berlin.de/Vorlesungen/SoSe04/KombGeoI/`

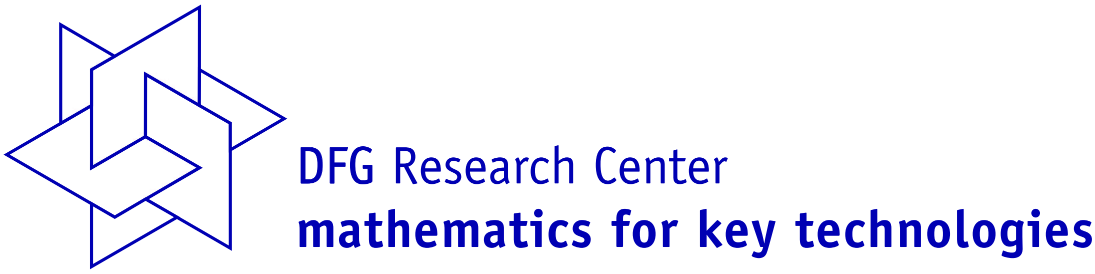
Sommer 2004
Stand 20. April 2004

First Problem Set 'Discrete Geometry'

Basic Notions and Examples of Polytopes

Homework

1. Find coordinates for the regular icosahedron.
   Hint:



5 points

2. Let $K \subseteq \mathbb{R}^d$. The equality

$$\operatorname{conv} K \quad = \quad \operatorname{aff} K \cap \operatorname{cone} K$$

doesn't hold in general. Which inclusion holds? Prove it. Where (exactly!) does the proof for the other inclusion fail? Give a counter-example! Under which extra assumptions does equality hold? Are these conditions necessary?

5 points

3. Classify all 3-dimensional 0/1-polytopes up to
   (a) congruence and
   (b) combinatorial equivalence.

   Hint: `polymake` may help your intuition. **5 points**

Σ 15 points

p.t.o

Further Material

1. Classify the 3-polytopes with $4, 5$ or 6 vertices up to combinatorial equivalence. Construct them using `polymake`.

2. Let $S \subseteq \mathbb{R}^n$. Fill in the following table, finding either proofs or counter-examples in $\mathbb{R}^2$.

If S is a ... then it is a ...	convex set	affine subspace	cone	$\mathcal{V}$-polyhedron	$\mathcal{V}$-polytope
convex set					
affine subspace					
cone					
$\mathcal{V}$-polyhedron					
$\mathcal{V}$-polytope					

   What needs to be adjusted in the proofs or counter-examples if $\mathcal{V}$ is replaced by $\mathcal{H}$?

3. If you are not familiar with the notion of affine independence you should carefully work through this exercise.
   Let $x_0, \ldots, x_n \in \mathbb{R}^d$. Show the equivalence of the following conditions.

   (a) $x_0, \ldots, x_n$ are affinely independent.
   (b) $x_1 - x_0, \ldots, x_n - x_0$ are linearly independent.
   (c) $\dim \operatorname{span}\{x_1 - x_0, \ldots, x_n - x_0\} = n$.
   (d) $\binom{x_0}{1}, \ldots, \binom{x_n}{1} \in \mathbb{R}^{d+1}$ are linearly independent.
   (e) Let $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$.
   If $\lambda_0 x_0 + \ldots + \lambda_n x_n = 0$ and $\lambda_0 + \ldots + \lambda_n = 0$,
   then $\lambda_0 = \ldots = \lambda_n = 0$.

4. Let P be a 3-dimensional polytope such that any two vertices are adjacent (joined by an edge). Show that P is a tetrahedron.

5. Any simple and simplicial polytope is a simplex or an n-gon.

6. Construct a 3-polytope which is combinatorially equivalent to a cube and such that the disjoint facets are in orthogonal planes.

7. Show:
   A polytope is the affine image of a crosspolytope if and only if it is centrally symmetric.

8. If $x, y \in M \implies \forall \lambda \in \mathbb{R} : \lambda x + (1-\lambda) y \in M$ then M is an affine subspace.

9. Realize the n-gon $C_2(n)$ with small integral coordinates in $\{0, 1, \ldots, f(n)\}$, for a 'small' $f(n)$.

10. Compute the number of facets and vertices of 'Complete Stacked' d-polytopes. Have a go on the complete f-vector!

11. Create a random 3-polytope with 100 vertices using `polymake`. Find a vertex of maximal degree and cut it off.