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Fakultät II – Institut für Mathematik Stand 27. April 2004
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`http://www.math.tu-berlin.de/Vorlesungen/SoSe04/KombGeoI/`

Second Problem Set 'Discrete Geometry'

Main Theorem and Lattices

Deadline: *Wednesday, 5 May 2004*

Don't miss the quiz! New on the webpage!
Identify the polytope and win a phantastic prize!

Homework

1. Show: The intersection of two faces of a polytope P is a face of P.

5 points

2. State and prove a Farkas lemma for systems of the form $Ax \leq z, x \geq 0$.

5 points

3. Let P be a d-polytope with n vertices and m facets. Number the vertices 1 to n and the facets 1 to m. Let $M(P) = (m_{ij}) \in \{0,1\}^{m \times n}$ be the facet-vertex-incidence-matrix of P, that means $m_{ij} = 1$ if and only if facet i contains vertex j (otherwise $m_{ij} = 0$).
   (a) How can you reconstruct the face lattice of P from $M(P)$ uniquely?
   (b) Where do you get the dimension of P in this calculation?
   (c) Where could the algorithm fail if the given matrix is not the facet-vertex-incidence-matrix of a polytope but for example of an unbounded polyhedron?
   (d) How are the matrices $M(P)$ and $M(P^{\Delta})$ related?

5 points

Σ 15 points

p.t.o.

Further Material

1. Let P be a 3-polytope with n facets. As we shall see P has at most $2n-4$ vertices. Use Fourier-Motzkin-elimination to obtain the polygon P'. How many vertices/facets has P' at most? Give for each $n \geq 4$ an example with the maximum number of vertices/facets.

2. Find a $\mathcal{V}$-description of the $\mathcal{H}$-polytope given by

$$0 \leq x_i \leq 1$$

for $i \in \{1, \dots, d\}$ and

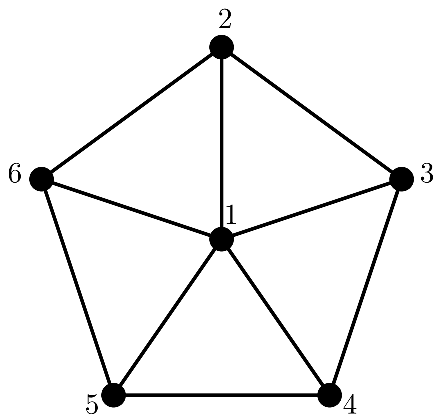
$$\sum_{i=1}^{d} x_i \leq \frac{3}{2}.$$

How many facets and vertices are there?

Remark. These polytopes are known as *dwarfed cubes.*

3. Show:
If a polytope is k-neighborly then every ℓ-face with $\ell \leq 2k-1$ is an ℓ-simplex. Conclude that every $(\lfloor \frac{d}{2} \rfloor + 1)$-neighborly polytope is a simplex.

4. Let $G = (V, E)$ be the following graph. (V denotes the set of vertices and E the set of edges.)



A subset V' of V is called *stable* if no edge in E joins two vertices in V'. The convex hull of all incidence vectors of stable sets is a $\mathcal{V}$-polytope in $\mathbb{R}^6$, which we denote by STAB(G). The aim of this exercise is the description of STAB(G) as an $\mathcal{H}$-polytope.

(a) Find all stable sets of the graph G and determine the dimension of STAB(G).

(b) Show that no inequality of the form $x_i + x_j \leq 1$ where ij is an edge of G, defines a facet of STAB(G).

(c) Show that every inequality of the form $x_i \geq 0$ where i is a vertex of G , defines a facet of STAB(G).

(d) Show that every inequality of the form $x_i + x_j + x_k \leq 1$ where ijk is a triangle of G , defines a facet of STAB(G).

(e) Show that the inequality $2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 2$, defines a facet of $\mathrm{STAB}(G)$.

(f) Show that there are no other facets of $\mathrm{STAB}(G)$.

(g) Verify this result by using `polymake` to find all facets of $\mathrm{STAB}(G)$.

(Hint for part (f): Begin by showing that every vertex of the polytope defined by the inequalities in (c), (d) and (e) has integer coordinates.)

5. Define the *face figure* P/F for any face of P by $P/F := (F^\diamond)^\Delta$, that is, the polar of the face of P^Δ which corresponds to F.

   (a) Show that P/F is a polytope of dimension

   $$\dim(P/F) = \dim(P) - \dim(F) - 1.$$

   (b) Characterize the face lattice of a face figure in terms of the face lattice of P and of the element $F \in L(P)$.

   (c) Describe a more direct construction of P/F, generalizing the case of a vertex figure.

   (d) How can the face figure P/F be obtained as an iterated vertex figure?

   Remark. The face figures P/F are also known as the *quotients* of P. Thus a quotient of P is the same thing as an iterated vertex figure.

6. Let $(S, \leq)$ be a finite poset. Show that any two of the following properties imply the third.

   (a) $(S, \leq)$ is bounded.

   (b) Any two elements $x, y \in S$ have a unique minimal upper bound $x \vee y$.

   (c) Any two elements $x, y \in S$ have a unique maximal lower bound $x \wedge y$.

7. Show:
   Every $\mathcal{V}$-polytope with n vertices is an affine image of the $(n-1)$-dimensional standard simplex.

8. (a) Let P be a 3-polytope and v a vertex of P. Show that the number of facets containing v equals the number of edges containing v.

   (b) Prove or disprove the claim of part (a) for polytopes of higher dimension.

   (c) Prove or disprove the claim of part (a) for *simple* polytopes of higher dimension.

9. Look at the d-cube $C_d \subseteq \mathbb{R}^d$.

   Find all $3^d + 1$ faces of C_d together with a (natural) bijection of the non-empty faces and the set of sign vectors $\{+, -, 0\}^d$.

10. Show that there are exponentially many combinatorial types of 3-dimensional stacked polytopes with n vertices.