Technische Universität Berlin Sommer 2004
Fakultät II – Institut für Mathematik Stand 10. Mai 2004
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http://www.math.tu-berlin.de/Vorlesungen/SoSe04/KombGeoI/

Fourth Problem Set 'Discrete Geometry'

4-Polytopes, Diagrams

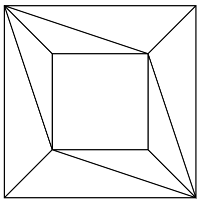
Deadline: *Wednesday, 19 May 2004*

Homework

1. Let P be a 4-polytope and u, v two vertices of P.
   Prove: There is an edge connecting u and v if and only if there are at least 3 facets containing both u and v.

5 points

2. (a) Find a small simplicial 2-diagram that isn't regular.
   Hint. Don't forget to prove that it works!
   (b) Show that the following 2-diagram is regular, but not Schlegel:



5 points

3. Describe how the Schlegel-diagram of a 3- or 4-polytope changes if you cut off a vertex.
   Hint. There are two cases!

5 points

Σ 15 points

p.t.o.

Further Material

1. How can stellar subdivisions be performed/visualized on Schlegel-diagrams?

2. If P has dimension at least 4, then the graph $G(P)$ is not planar. In fact, show that it contains a subdivision of the complete graph K_{d+1}.

   K_d is the complete graph on d vertices. A graph G *contains* a *subdivision* of a graph H, if you can get G from H by a series of the following operations:

   (a) Remove a vertex (and all incident edges),

   (b) Remove an edge,

   (c) Replace a path $v_0, \dots, v_k$ by an edge $v_0 v_k$.

   Hint. Induction and vertex figures.

3. A d-polytope P is called *dimensionally ambiguous* if there is a polytope Q of a different dimension $\dim(Q) \neq \dim(P)$ which has an isomorphic graph, $G(P) \cong G(Q)$.

   (a) Show that the d-simplex is dimensionally ambiguous for $d \geq 5$, but not for $d \leq 4$.
   (b) Show that 3-polytopes, and simple 4-polytopes, cannot be dimensionally ambiguous.
   Hint. Use Exercise 2.
   (c) Show that the d-cubes are dimensionally ambiguous for $d \geq 5$. For example, if Q is the standard 2-cube (also known as square), then the 4-polytope

   $$\operatorname{conv}(Q \times 2Q \cup 2Q \times Q)$$

   has a graph that is isomorphic to $G(C_5)$.

4. Construct a Schlegel diagram and calculate the f-vectors for

   (a) the pyramid over a cube.

   (b) the prism over an octahedron.

   (c) the product

   $$\Delta_2 \times \Delta_2$$

   and its polar.

   (d) the cyclic polytopes $C_3(6)$ and $C_4(7)$.

   You can do this exercise by hand and/or with `polymake`.

5. Let $P \subseteq \mathbb{R}^d$ be a d-polytope, $A \subseteq \mathbb{R}^d$ an affine k-subspace.
   Then $P \cap A$ is a polytope of dimension at most k.
   All faces of $P \cap A$ are of the form $F \cap A$ for faces $F \subset P$, and
   $$\dim F \ \geq \ \dim(F \cap A).$$