Technische Universität Berlin Sommer 2004
Fakultät II – Institut für Mathematik Stand 17. Mai 2004
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`http://www.math.tu-berlin.de/Vorlesungen/SoSe04/KombGeoI/`

Fifth Problem Set 'Discrete Geometry'

Shellings and Miscellaneous

Deadline: *Wednesday, 26 May 2004*

Homework

1. **5 points**
2. **5 points**
3. **5 points**

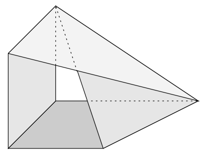
Σ **15 points**

Further Material

1. Show that the d-dimensional cube $C_d := [0,1]^d$ has the following properties:
   (a) Every triangulation of C_d without new vertices has at most $d!$ simplices of dimension d.
   (Hint: What can you say about the volume of the d-simplices of a triangulation of C_d without new vertices?)
   (b) The upper bound of part (a) is sharp, i.e. for every dimension d there is a triangulation of C_d with exactly $d!$ simplices of dimension d.
   (Hint: Show that there is a trianguletion of C_d that contains the simplex $\{(x_1,\ldots,x_d) \in \mathbb{R}^d : 0 \leq x_1 \leq \ldots \leq x_d \leq 1\}$.)

2. Given finite graphs G and H, we define that G is an *induced subgraph* of H if we can obtain a graph isomorphic to G by deleting a set S of vertices (and all edges adjacent to them) from H. We say that H is a *suspension* of G if additionally we require that the vertices of S are connected to all other vertices of H.
   (a) Show that every finite graph is an induced subgraph of the graph of a 4-polytope.
   (If G has $n \geq 5$ vertices, then start with $C_4(n)$, and introduce an extra vertex beyond every edge that is missing in G.)
   (b) For every finite graph there is some suspension which is the graph of a d-polytope, for some d.
   (If G has $n \geq 5$ vertices, then start with $C_4(n)$, and introduce an extra dimension and two new vertices for every edge in G that is supposed to be missing.)
   (c) Give an example of a 4-connected graph on $n \geq 5$ vertices which is not the graph of a 4-polytope. Can you construct a 4-regular graph with these properties?

3. Show that the subdivision of the *Möbius strip* drawn below with 6 vertices, 10 edges, and 4 facets can be realized as a polyhedral complex in $\mathbb{R}^3$, but not as a subcomplex of any polytope.



   This shows that there are polyhedral complexes that are not subcomplexes of polytopes.

4. Show that a set of facets of the d-cube determines a shellable subcomplex of ∂C_d if and only if it contains no facets (is empty), or all facets, or if it contains at least one facet such that the opposite facet is not in the complex.
   Deduce that the boundary complexes of the d-cubes are extendably shellable.
   How many shellings of the d-cube are there?

5. Describe a shelling of the d-dimensional crosspolytope. Use it to compute the f-vector and the h-vector of the d-dimensional crosspolytope.
   How many shellings of the octahedron are there? How many are there for the d-crosspolytope?

6. Given the h-vector of a simplicial polytope P, how can one derive the h-vector of the bipyramid bipyr(P)?

7. Let P be a polytope. Describe how the face lattice $\mathcal{L}(P)$ changes when one 'pulls a vertex'.

8. Glue two d-connected graphs by identifying d vertices of the first one with d vertices of the second one in pairs.
   Show that the result is again d-connected.

9. Characterize the f-vectors of cubical 3-polytopes.
   Hint. You may use that there is no cubical 3-polytope 9 vertices.
   You can earn **5 extra points** by proving this.

10. Prove: A graph G (viewed as a 1-dimensional simplicial complex) is shellable if and only if it is connected.

11. Every shelling $F_1, F_2, \ldots, F_s$ of the facets of a polytope P also induces, for every facet F_i, an ordering of the facets of F_i. Namely, one can take facets of F_i in the order in which they appear in the list $F_1 \cap F_i, \ldots, F_{i-1} \cap F_i, F_{i+1} \cap F_i, \ldots, F_s \cap F_i$. A shelling is *perfect* if this ordering is a shelling order of the boundary of F_i, for all i.
    For example, shellings of simplicial polytopes are always perfect.

(a) Show that there are shellings that are not perfect.

(b) Show that Bruggesser-Mani shellings are not perfect in general.

(c) Show that the d-cubes C_d have perfect shellings, for all $d \geq 1$.

(d) Show that all 3-polytopes have perfect shellings.

(e) Show that the polars of cyclic polytopes $C_d(n)^\Delta$ have perfect shellings.