Prof. Dr. Dr. h.c. Martin Grötschel
Andreas Bley
Benjamin Hiller

Exercise sheet 11

Deadline: Thursday, July 12th, 2007, 08:30 h in MA-313

Exercise 1: **4 points**

Let E be a finite set and $E_1, \ldots, E_k$ a partition of E. Furthermore, let $b_1, \ldots, b_k$ be integers satisfying $1 \leq b_i \leq |E_k|$. Then $(E, \mathcal{I})$ with $\mathcal{I}$ defined by

$$\mathcal{I} := \{I \subseteq E \mid |I \cap E_i| \leq b_i, 1 \leq i \leq k\}$$

is a matroid, called the *partitioning matroid*.

Determine the facets of the partitioning matroid based on the characterization of the facets of the matroid polytope given in the lecture.

Exercise 2: **4 points**

a) Let $b_1, \ldots, b_m$ be positive integers. Show that $L(a) = L(b_1, \ldots, b_m)$, where $a = \gcd(b_1, \ldots, b_m)$ is the greatest common divisor of $b_1, \ldots, b_m$.

b) Give an algorithm that from a set of possibly dependent vectors $a_1, \ldots, a_m \in \mathbb{Z}^n$ finds a basis $b_1, \ldots, b_k$ of $L(a_1, \ldots, a_m)$ such that $b_i^T e_j = 0$ for $j < i$, where e_j is the jth unit vector.

   Hint: Consider one dimension at a time and use the first part of this exercise.

Exercise 3: **1+2+1 points**

A matrix $A \in \mathbb{Z}^{m \times n}$ of full row rank is said to be in *integer normal form* if it is of the form $[B, 0]$, where $B \in \mathbb{Z}^{n \times n}$ is invertible and lower triangular. For every matrix $A \in \mathbb{Z}^{m \times n}$ of full row rank there is a unimodular matrix U such that AU is in integer normal form.

Prove the following theorem:

Theorem 1 *Let $A \in \mathbb{Z}^{m \times n}$ be a matrix of full row rank and let $[B, 0] = AU$ be the integer normal form of A with a unimodular matrix U. Let $b \in \mathbb{Z}^m$ and $\mathcal{F} = \{x \in \mathbb{Z}^n \mid Ax = b\}$.*

a) $\mathcal{F}$ is nonempty if and only if $B^{-1}b \in \mathbb{Z}^m$.

b) If $\mathcal{F} \neq \emptyset$, every element of $\mathcal{F}$ is of the form

$$x = U_1 B^{-1} b + U_2 z, z \in \mathbb{Z}^{n-m},$$

where U_1, U_2 are submatrices of U such that $U = [U_1, U_2]$.

c) $\mathcal{L} = \{x \in \mathbb{Z}^n \mid Ax = 0\}$ is a lattice and the column vectors of U_2 constitute a basis of $\mathcal{L}$.

Exercise 4: **4 points**

The theorem of the last exercise suggests a way to solve certain integer programs via alternative bases of lattices. Consider the integer program

$$\begin{aligned} \max \quad & c^T x \\ & x \in \mathcal{F} := \{x \in \mathbb{Z}^n_{\geq 0} \mid Ax = b\}, \end{aligned}$$

where A is an integer matrix of full row rank and b and c are integer vectors. Suppose x_0 is an integer point satisfying $Ax_0 = b$. Then every $x \in \mathcal{F}$ can be written as

$$x = x_0 + y, \quad \text{for some } y \in \mathbb{Z}^n \text{ s. t. } \; Ay = 0, y \geq -x_0.$$

Let $\mathcal{L} := \{y \in \mathbb{Z}^n \mid Ay = 0\}$ and consider the integer normal form $[B, 0]$ of A obtained using the unimodular matrix U. Due to b) in the theorem, we have

$$\mathcal{L} = \{y \in \mathbb{Z}^n \mid y = U_2 z, z \in \mathbb{Z}^{n-m}\},$$

which allows us to reformulate the original IP as

$$\begin{aligned} \max \quad & c^T U_2 z \\ & U_2 z \geq -x_0, \\ & z \in \mathbb{Z}^{n-m}. \end{aligned}$$

Since all bases of $\mathcal{L}$ can be obtained from U_2 via unimodular matrices, we get an alternative reformulation for any basis B of $\mathcal{L}$, namely

$$\begin{aligned} \max \quad & c^T B z \\ & B z \geq -x_0, \\ & z \in \mathbb{Z}^{n-m}. \end{aligned}$$

For a suitable chosen basis the reformulation might be easier to solve than the orginal formulation.

Solve the following IP via reformulations based on alternative bases.

$$\begin{aligned} \max \quad & x_1 + x_2 + x_3 \\ & 3x_1 + 7x_2 + 10x_3 = 19, \\ & x_1, x_2, x_3 \in \mathbb{Z}_{\geq 0}. \end{aligned}$$