TU Berlin
Institut für Mathematik
Dr. Britta Peis
Dr. Sebastian Stiller
Jannik Matuschke

Integer Linear Programming
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# Assignment 1

## Exercise 1

Formulate the following combinatorial problems as integer linear programs.

a) *Partition*: Given $n$ numbers $a_1, \ldots, a_n \in \mathbb{N}$ determine whether there exists a subset $S \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in S} a_i = \sum_{i \in \{1,\ldots,n\} \setminus S} a_i$.

b) *Satisfiability*: Given $m$ clauses $C_1, \ldots, C_m$ over $n$ boolean variables $x_1, \ldots, x_n$, find a truth assignment to the variables that satisfies all clauses. (A clause is a disjunction of literals, e.g. $C_3 = x_2 \vee \neg x_5 \vee x_{13}$.)

c) *Integral Max Flow*: Given an $s$-$t$-network with capacities $u : E \to \mathbb{Z}$, find an integral $s$-$t$-flow of maximum value.

What complexity results can be derived from the above modelling?

## Exercise 2

a) Let $A \in \mathbb{Q}^{m \times n}$. Show that the encoding size of $\det(A)$ is bounded by a polynomial in the encoding size of $A$.

b) Show that *Linear Algebra* has a good characterization, i.e. for each instance there either exists a certificate for the existence or for the non-existence of a solution $x \in \mathbb{Q}^n$ to $Ax = b$ and the size of this certificate is bounded by a polynomial in the input size. *Hint:* You already learned about the certificates in the lecture. Use part a) to show that they are sized polynomially.

*Reminder:* The encoding size of a rational number $r = \frac{p}{q}$ with $p \in \mathbb{Z}, q \in \mathbb{N}$ coprime is

$$\langle r \rangle := 1 + \lceil \log_2(|p| + 1) \rceil + \lceil \log_2(q + 1) \rceil$$

and the encoding size of a matrix $A \in \mathbb{Q}^{m \times n}$ is

$$\langle A \rangle := mn + \sum_{i \in [m]} \sum_{j \in [n]} \langle a_{ij} \rangle.$$

## Exercise 3

Draw the lattice that is generated by the columns of the matrix

$$A = \begin{pmatrix} 2 & 4 & 0 \\ 5 & -1 & 3 \end{pmatrix}.$$

*Hint:* It might be helpful to determine a basis of the lattice first.

**Exercise 4**

Determine an integral solution of the diophantine equation $15x_1 + 51x_2 + 18x_3 = b$ or show that none exists for

a) $b = 17$.

b) $b = 42$.

**Exercise 5**

Let

$$A = \begin{pmatrix} 2 & 4 & 2 & -6 \\ 3 & 10 & 7 & -13 \\ 6 & 0 & -1 & -16 \end{pmatrix}.$$

a) Determine a basis of the lattice spanned by the columns of $A$.

b) Determine all integral solutions of the system of diophantine equations $Ax = b$ or show that none exists for

   i) $b = \begin{pmatrix} 4 \\ 2 \\ 9 \end{pmatrix}$.

   ii) $b = \begin{pmatrix} 2 \\ 4 \\ 9 \end{pmatrix}$.