Technische Universität Berlin **Sommersemester 2011**
Fakultät II – Mathematik und Naturwissenschaften
Institut für Mathematik
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Numerische Mathematik II/ Numerical Analysis II
1. Assignment

Homework: HW1 (Due 26.04.2011/27.04.2011)

1. Solve the following ODEs

   (a) $$y'(t) = 2y(t) - e^t, \quad y(0) = 2.$$

   (b) $$\begin{aligned} y_1'(t) &= 2y_1(t) - y_2(t), & y_1(0) = 1, \\ y_2'(t) &= -y_1(t) + 2y_2(t), & y_2(0) = 0. \end{aligned}$$

2. Determine whether the following initial value problem has a unique solution.

   $$y' = y^{2/3}, \quad y_0 = 0.$$

   If not, discuss which condition in the Picard-Lindelöf Theorem is not satisfied and why.

3. Give the geometric illustration of the modified Euler method (known also as Collatz method) and prove that it is of order $p = 2$ (Konsistenzordnung $p = 2$).

4. Check that an explicit two-stage Runge-Kutta method has order $p = 2$ (Konsistenzordnung $p = 2$) if

   $$\gamma_1 + \gamma_2 = 1, \ \gamma_2\alpha_2 = 1/2, \text{ and } \gamma_2\beta_{2,1} = 1/2$$

   Where $\alpha_2, \beta_{2,1}, \gamma_1, \gamma_2$ are the coefficients of the Butcher array

   $$\begin{array}{c|cc} 0 & & \\ \alpha_2 & \beta_{2,1} & \\ \hline & \gamma_1 & \gamma_2 \end{array} .$$

5. Determine all Runge-Kutta methods of order $p = 2$ (Konsistenzordnung $p = 2$) of the form

   $$\begin{array}{c|ccc} 0 & & & \\ c_2 & c_2 & & \\ c_3 & 0 & c3 & \\ \hline & 0 & 0 & 1 \end{array} .$$

Programming assignment: PA1 (Due 03.05.2011/04.05.2011)

1. Write a program that solves the ODE $y'(t) = f(t, y), \ y(t_0) = y_0$ using

   - explicit Euler method,
   - modified Euler method,
   - Heun method,
   - classical Runge-Kutta method

   on the interval $[t_0, t_0 + a]$.

(a) The program should be called with

```
[h,t,u] = exprk(f,phi*,t0,y0,N,a)
```

where `t0` $= t_0$ is the beginning of the interval, `y0` $= y_0$ the initial value, `N` $= N$ total number of steps and `a` $= a$ the interval length. The output are `h` $= h = a/N$ the step size, `u` $= \mathbf{u} = [u_1, \ldots, u_n]$ the numerical approximation of $y(t)$ at points `t` $= [t_0, \ldots, t_N]$.

(b) The general one-step method can be defined as

$$\begin{aligned} u_0 &= y_0 \\ u_{i+1} &= u_i + h\phi(t_i, u_i, h, f) \\ t_i &= t_0 + ih \end{aligned}$$

where $\phi(t_i, u_i, h, f)$ is known as the *increment function*. In order to make the program more modular and easier to maintain for each of the methods write an appropriate increment function $\phi(t, u, h, f)$, i.e.,

```
[phi] = phiexpeul(t,u,h,f)
[phi] = phimodeul(t,u,h,f)
[phi] = phiheun(t,u,h,f)
[phi] = phirk(t,u,h,f)
```

with `t,u` being appropriate vectors. To determine the right hand side $f(t, y)$ the subroutines `phi*` may call

```
[yp] = f(t,y)
```

2. Test your program for the problems from Exercise 1 with $a = 1$ and $N = 10, 100, 1000$.

(a) Plot the numerical solution against the exact solution for each step $t_i = ih$ and $N = 10, 100, 1000$.

(b) Plot the error $\mathtt{e}_i = \|y(t_i) - \mathtt{u}_i\|_\infty$ for each step $t_i = ih$ and $N = 10, 100, 1000$. Plot the error also in the logarithmic scale.

(c) Summarize the numerical results, i.e., $y(t_i)$, $\mathtt{u}_i$, $\mathtt{e}_i$ for steps $t_0, t_0 + 0.2, t_0 + 0.4, t_0 + 0.6, t_0 + 1.0$ and $N = 10, 100, 1000$ in a table.