Technische Universität Berlin **Sommersemester 2011**
Fakultät II – Mathematik und Naturwissenschaften
Institut für Mathematik
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Numerische Mathematik II/ Numerical Analysis II
3. Assignment

Exercise: EX3 (3-4.05.2011)

1. Consider the multi-step method

$$u_{j+1} = \alpha_0 u_j + h(\beta_1 f_{j+1} + \beta_0 f_j)$$

where $f_j = f(t_j, u_j)$. Find the parameters $\alpha_0, \beta_1, \beta_0$ such that the local truncation error is minimal and determine the order of the method. What else you can say about this method?

2. Determine the order of the following multi-step method

$$u_{j+3} - u_{j+1} = h\Big(\frac{7}{3}f_{j+2} - \frac{2}{3}f_{j+1} + \frac{1}{3}f_j\Big).$$

Homework: HW3 (10-11.05.2011)

1. Consider the multi-step method **(5 pts.)**

$$u_{j+3} = \alpha_0 u_j + \alpha_2 u_{j+2} + h(\beta_1 f_{j+1} + \beta_2 f_{j+2} + \beta_3 f_{j+3})$$

where $f_i = f(t_i, u_i)$. Find the parameters $\alpha_0, \alpha_2, \beta_1, \beta_2, \beta_3$ such that the local truncation error is minimal and determine the order of the method. What else you can say about this method?

2. Determine the order of the following multi-step method **(5 pts.)**

$$u_{j+3} - u_{j+2} = h\Big(\frac{3}{8}f_{j+3} + \frac{19}{24}f_{j+2} - \frac{5}{24}f_{j+1} + \frac{1}{24}f_j\Big).$$

3. Let y be the solution of the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$, where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a global Lipschitz condition in the second argument, i.e.,

$$|f(t,x) - f(t,y)| \leq L|x - y|$$

for all $t, x, y \in \mathbb{R}$. Furthermore, let the *consistent* k-step method

$$\sum_{l=0}^{k} \alpha_l u_{j+l} = h \sum_{l=0}^{k} \beta_l f(t_{j+l}, u_{j+l})$$

with $\alpha_k \neq 0$ be given.

(a) Show, that the local discretization error $\tau(t, h)$ satisfies

$$\tau(t,h) = \omega(h),$$

where $\omega(h) \to 0$ for $h \to 0$. **(5 pts.)**
(*Hint:* You are not allowed to assume that f is differentiable, so a Taylor expansion is out of question. Instead, you may use that y satisfies an integral equation and you may want to apply the *mean value theorem for integration* (Mittelwertsatz der Integralrechnung).)

(b) Let $y_l = y(t_l)$ with $t_l = t_0 + lh$ for $l = 0, \ldots, k$. Furthermore, let u_k be the solution of the k-step method with exact initial values $y_0, \ldots, y_{k-1}$. Show that

$$|u_k - y_k| \leq h \cdot \widetilde{\omega}(h),$$

where $\widetilde{\omega}(h) \to 0$ for $h \to 0$. **(5 pts.)**

(*Hint:* Use part (a).)

Programming assignment: PA3 (17-18.05.2011) **(10 pts.)**

1. Write a program that solves the ODE $y'(t) = f(t,y)$, $y(t_0) = y_0$ using the predictor-corrector methods $P(EC)^{m_0}E$ and $P(EC)^{m_0}$ on the interval $[t_0, t_0 + a]$. Use the ABM-method with the 3-step Adams-Bashforth method as predictor and 3-step Adams-Moulton method as corrector, i.e., use the pair of methods:

$$\begin{aligned} u_{j+3} &= u_{j+2} + \tfrac{h}{12}\Big(23f_{j+2} - 16f_{j+1} + 5f_j\Big) && \text{Adams-Bashforth} \\ u_{j+3} &= u_{j+2} + \tfrac{h}{24}\Big(9f_{j+3} + 19f_{j+2} - 5f_{j+1} + f_j\Big) && \text{Adams-Moulton} \end{aligned}$$

   Use the classical Runge-Kutta method of order 4 to obtain the necessary initial values.

2. Test your program for the problems

   (a)
   $$y'(t) = 2y(t) - e^t, \quad y(0) = 2.$$

   (b)
   $$\begin{aligned} y_1'(t) &= 2y_1(t) - y_2(t), & y_1(0) = 1, \\ y_2'(t) &= -y_1(t) + 2y_2(t), & y_2(0) = 0. \end{aligned}$$

   with $N = 5, 10, 100, 1000$ and $a = 1$ and $m_0 = 0, \ldots, 5$, where m_0 is the number of corrector iterations.

3. Compare the results with exact solutions and approximations obtained with the classical Runge-Kutta method. Plot the exact solutions, the approximate solutions by Adams-Bashfort&Adams-Moulton, and the approximations obtained with the classical Runge-Kutta method. Plot also the corresponding errors $\mathrm{e}_i = \|y(t_i) - \mathrm{u}_i\|_\infty$.

4. For steps $t_0, t_0 + 0.2, t_0 + 0.4, \ldots$ summarize the numerical examples in a table, i.e., exact solutions (Ex), Adams-Bashfort & Adams-Moulton (AdBaAdMo,m_0), and classical Runge-Kutta (clRK).

Hint:

`exprk` - the classical Runge-Kutta method (see programming assignment PA1),

`adbaadmo` - the predictor-corrector method with Adams-Bashfort (predictor) and Adams-Moulton (corrector), call `exprk` to obtain initial values, input parameters should be f, t_0, a, y_0, N, m_0, output should be the approximation $(u_i)_i$ and the corresponding time points $(t_i)_i$.