Technische Universität Berlin **Sommersemester 2011**
Fakultät II – Mathematik und Naturwissenschaften
Institut für Mathematik
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Numerische Mathematik II/ Numerical Analysis II
5. Assignment

Homework: HW5 (24-25.05.2011)

1. Check the zero-stability of the following methods **(2 pts.)**
   (a) $u_{j+2} = 3u_{j+1} - 2u_j - hf_j$
   (b) $u_{j+2} = u_{j+1} + \frac{h}{2}(3f_{j+1} - f_j)$ (second-order Adams-Bashforth)
2. Determine the characteristic polynomial $\rho(z)$ for all zero-stable four step methods ($k = 4$) of order $p = 6$. Assume that $\alpha_k = 1$. **(4 pts.)**
3. Let the Cayley transformation be defined as
$$C(\eta) = \frac{1+\eta}{1-\eta} = z.$$
   (a) Show that the inverse of $C(\eta)$ is defined as **(1 pt.)**
$$C^{-1}(z) = \frac{z-1}{z+1} = \eta.$$
   (b) Show that $\text{Re}(\eta) = 0 \Rightarrow |z| = 1$ and $\text{Re}(\eta) < 0 \Rightarrow |z| < 1$. **(2 pts.)**
   (c) Show that if $z = -1$ is a simple root of $\rho(z)$ then deg $R(\eta) = k - 1$. **(2 pts.)**
4. Assume that the multi-step method
$$\sum_{l=0}^{k} \alpha_l u_{j+l} = h \sum_{l=0}^{k} \beta_l f(t_{j+l}, u_{j+l}),$$
   satisfies $\frac{\beta_k}{\alpha_k} \leq 0$. Show that the method is then of order $p \leq k$.
   Do this as follows:
   Using the proof of the theorem of Dalquist and definitions of $R(\eta)$ and $S(\eta)$ from the lecture
   (a) show that $p > k$ implies **(2 pts.)**
$$S(\eta) = R(\eta)\Big(\frac{1}{2\eta} + \sum_{j=1}^{k} \mu_j \eta^{j-1}\Big) - \sum_{j=1}^{k-1} \Big(\sum_{s=j}^{k-1} a_{s+1}\mu_{k+j-s}\Big)\eta^{k+j},$$
   (b) show that **(1 pt.)**
$$R(1) = 2^{-k}\alpha_k, \quad S(1) = 2^{-k}\beta_k \quad \text{and} \quad \sum_{j=1}^{\infty} \mu_j = -\frac{1}{2},$$
   (c) consider the limit $\eta \to 1$ for the formula in (a) and conclude that this leads to a contradiction. **(1 pt.)**

Programming assignment: PA4 (31.05.2011 - 01.06.2011) **(10 pts.)**

1. Write the program `msv.m` which solves the IVP

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0 \quad \text{in } [t_0, t_0 + a]$$

with $y, f \in \mathbb{R}^n$ using the (**explicit**) linear multi-step method given by parameters α_i and β_i. Use the classical Runge-Kutta method for determining a sufficient number of starting values. Moreover write the program `stabregion.m`, which for the given α_i and β_i determines the region of absolute stability.

Analyse the behaviour of the solution of the IVP

$$y' = -8y, \quad y(0) = 2, \quad t \in [0, 1]$$

depending of the choice of α_i and β_i for $N = [10, 20, 40, 80, 160]$.

Compare four different multi-step methods

(a) $u_{j+1} = u_j + hf_j$ (explicit Euler)

(b) $u_{j+2} = u_j + 2hf_{j+1}$ (2-step Nyström)

(c) $u_{j+2} = u_{j+1} + \frac{h}{2}[3f_{j+1} - f_j]$ (2-step Adams-Bashforth)

(d) $u_{j+4} = u_{j+3} + \frac{h}{24}[55f_{j+3} - 59f_{j+2} + 37f_{j+1} - 9f_j]$ (4-step Adams-Bashforth)

Plot the exact solution and the corresponding approximations and the region of absolute stability.

(**Note: Choose some interesting examples, plot the results and comment on them. Do not plot all the results for all possible parameters.**)

Hint for the structure of the program:

Write two routines

(a) `msv` - for solving the IVP using a linear multi-step method, in the form

`[h,t,u] = msv(fun,t0,u0,N,a,alpha,beta)`,

where `fun` is the name of the routine which determines the right-hand side $f(t, y)$ of the ODE, `t0` starting point, `u0` initial value, `N` number of steps, `a` interval length and `alpha=[alpha_0,...,alpha_k]`, `beta=[beta_0,...,beta_k(=0)]` are the coefficients of the linear multi-step method. The routine should return the step-size `h=a/N`, the interval lattice `t` $= [t_0, t_0 + h, ..., t_0 + Nh]$ and the approximate solution `u` $= [u_0, u_1, ..., u_N]$.

(b) `stabregion` - to determine the region of absolute stability, in the form

`stabregion(alpha,beta,scale,stepre,stepim)`,

where `alpha=[alpha_0,...,alpha_k]`,`beta=[beta_0,...,beta_k(=0)]` are the coefficients of the linear multi-step method, `scale=[realmin,realmax,imagmin,imagmax]` the region in which the region of stability should be determined and `stepre` or `stepim` the step-size along the real or imaginary axis for presenting the region of absolute stability. (In order to compare the given methods choose `scale`$= [-3, 3, -3, 3]$ and `stepre`$=$ `stepim`$= 0.006$.) If the method is absolutely stable in $(\mathrm{Re}(h\lambda), \mathrm{Im}(h\lambda))$ plot this point in red. Otherwise plot it in white.

Note: To determine the zeros of a polynomial use the Matlab function `roots` (see help).