Technische Universität Berlin **Sommersemester 2011**
Fakultät II – Mathematik und Naturwissenschaften
Institut für Mathematik
Prof. Dr. Christian Mehl, Dr. Agnieszka Miedlar Stand: June 7, 2011

Numerische Mathematik II/ Numerical Analysis II
8. Assignment

Homework: HW8 (21-22.06.2011)

1. Calculate the nodal FEM basis functions $\varrho_1, \dots, \varrho_5$ which span the space of quartic polynomials. Use the nodes $-1, -\frac{1}{2}, 0, \frac{1}{2}, 1$. **(3 pts.)**

2. Let the following boundary value problem be given

$$-y''(x) + \pi^2 y(x) = 2\pi^2 \sin(\pi x), \quad y(0) = y(1) = 0$$

with the exact solution $y(x) = \sin(x)$.

   (a) Use the Galerkin method to determine analytically the approximation of the solution in space S being the largest possible subspace of Π_3, i.e., the space of polynomials with degree at most three. Do not forget to include the boundary conditions. In order to do that determine the basis functions for S and write the corresponding linear system. Calculate the solution and write it in the form

$$P(x) = a_3x^3 + a_2x^2 + a_1x + a_0.$$

   Plot the solution and compare it with the exact solution.
   Note: You may use symbolic packages like Matlab or Mathematica. **(3 pts.)**

   (b) What approximations you will get if you use the Galerkin method with space

$$S_n = \{\sin(k\pi x) \mid k = 1, \dots, n\},$$

   for $n \leq 1$? **(2 pts.)**

3. Let V be a real, in general not finite dimensional, vector space with th inner product $[\cdot, \cdot]$. Furthemore let

$$\ell : V \to \mathbb{R}$$

be a continous linear mapping ($\ell \in V^*$).
Show that

$$F(y) = [y, y] - 2\ell(y)$$

takes its minimum for y if and only if

$$[y, v] = \ell(v) \text{ for all } v \in V.$$

Moreover, there is at most one minimum.
Hint: Use

$$\varphi(\varepsilon) = F(y + \varepsilon v), \quad \varepsilon \in \mathbb{R}, \; v \in V.$$

(4 pts.)

4. Consider the following boundary value problem

$$\left.\begin{aligned} -u''(x) + \beta u(x) &= f(x) \quad \text{in } \Omega := (0,1) \subset \mathbb{R}, \\ u(0) &= u(1) = 0, \end{aligned}\right\} \qquad (1)$$

with $\beta \in \mathbb{R}$, $\beta \geq 0$.

(a) Let $V = C_0^1(\overline{\Omega}) := \{v \in C^1(\overline{\Omega}) \mid v(0) = v(1) = 0\}$. Determine the symmetric bilinear form $a : V \times V \to \mathbb{R}$ and the linear functional $\ell : V \to \mathbb{R}$ such that (1) can be written in the variational formulation

$$\text{Find } u \in V \text{ mit } a(u,v) = \ell(v) \text{ for all } v \in V.$$

(3 pts.)

(b) Let $N \in \mathbb{N}$ and $0 = x_0 < x_1 < \ldots < x_N < x_{N+1} = 1$ be a grid (mesh) defined on $\overline{\Omega}$ with the grid-sizes (mesh-sizes) $h_i := x_i - x_{i-1}$ for $i \in \{1, \ldots, N+1\}$. Consider elements $\overline{\Omega}_i := [x_{i-1}, x_i]$ for $i \in \{1, \ldots, N+1\}$ and the finite element space

$$V_h^1 := \{\varphi \in C^0(\overline{\Omega}) \mid \varphi|_{\Omega_i} \text{ is linear } \quad \forall i \in \{1, \ldots, N+1\}, \ \varphi(0) = \varphi(1) = 0\}$$

with shape functions ϱ_i, $i = 1, \ldots, N$ defined as

$$\varrho_i(x) = \begin{cases} 0 & \text{if} \quad x < x_{i-1}, \\ \frac{x - x_{i-1}}{h_i} & \text{if} \quad x_{i-1} \le x < x_i, \\ \frac{x_{i+1} - x}{h_{i+1}} & \text{if} \quad x_i \le x < x_{i+1}, \\ 0 & \text{if} \quad x_{i+1} \le x. \end{cases}$$

With a and ℓ defined before determine the matrix $A \in \mathbb{R}^{N \times N}$ (called stiffness matrix) and the vector $b \in \mathbb{R}^N$ (called load vector) for the Galerkin approximation $u_h = \sum_{i=1}^N c_i \varrho_i \in V_h^1$, where $c = (c_1, \ldots, c_N)^\top$ is a solution of the linear system $Ac = b$. The entries of matrix A can be determined explicitely by calculating the appropriate integrals.

(4 pts.)

Programming assignment: PA8 (28-29.06.2011) **(10 pts.)**

Using Exercise 4 write a program `FEM1d.m` which solves the boundary value problem (1) using the 1D finite element method.

1. The routine `FEM1d.m` should be of the following form

   `[c,T,h] = FEM1d(beta,f,T),`

   where `beta`, `f` are the parameter and the right side function defining the boundary value problem, `T`$=[x_1, \ldots, x_N]$ is the grid defined on $\overline{\Omega}$, `N` the number of grid points. Note that `T`$= [x_1, \ldots, x_N]$ does not have to be an equidistant grid, so the size h_i of each element $\overline{\Omega}_i := [x_{i-1}, x_i]$ may be different. The output will be a vector `c`, grid `T` and vector `h` with sizes of elements $\overline{\Omega}_i := [x_{i-1}, x_i]$. For the equidistant grid `T` all the entries in vector `h` will be the same.

2. Since the program should work for every function `f` you should use the Trapezoidal rule to calculate the integrals needed to determine the load vector `b`. Please note that the basis functions have a small support, each of them is not zero only on two neighbouring elements.

3. Test your program for the following problems

   (a) $\beta = 0$ and $f(x) = 1$. The exact solution is given by $u(x) = \frac{1}{2}x(1 - x)$.

   (b) $\beta = 1$ and $f(x) = e^x 4x$. The exact solution is given by $u(x) = e^x(x - x^2)$.

4. Write the Matlab script `main.m` (takes no input arguments) which presents the numerical results: grid point x_i, approximate solution c_i, the exact solution $u(x_i)$ in a table. In addition plot the error against the mesh size, i.e., $\|c - u\|_\infty$ for the approximations obtained on the equidistant grid with mesh sizes $h_p = \frac{1}{2^p+1}$, for $p = \{1, \ldots, 14\}$. Hint: Make these plots using `loglog` scale where on one axis you have different values of h_p and the error on the other.