Technische Universität Berlin **Sommersemester 2011**
Fakultät II – Mathematik und Naturwissenschaften
Institut für Mathematik
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Numerische Mathematik II/ Numerical Analysis II
9. Assignment

Homework: HW9 (28-29.06.2011)

1. Let $U^TAV = \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_p)$, $p = \min\{m, n\}$ be the singular value decomposition of $A \in \mathbb{R}^{m \times n}$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_p = 0$. Show that
   (a) $\text{rank}(A) = r$;
   (b) $\text{Ker}(A) = \text{span}\ \{v_{r+1}, \ldots, v_n\}$;
   (c) $\mathcal{R}(A)(= \text{Image(A)}) = \text{span}\ \{\text{u}_1, \ldots, \text{u}_\text{r}\}$.

   (3 pts.)

2. Let the matrix $A = U + iV \in \mathbb{C}^{n \times n}$, $U, V \in \mathbb{R}^{n \times n}$, be Hermitian. Moreover, let
$$B = \begin{bmatrix} U & -V \\ V & U \end{bmatrix}.$$
   Show that
   (a) U is symmetric, V is skew-symmetric and B is symmetric.
   (b) For $u, v \in \mathbb{R}^n$ let
$$z = \begin{bmatrix} u \\ v \end{bmatrix} \text{ and } z' = \begin{bmatrix} -v \\ u \end{bmatrix}$$
   and $\lambda \in \mathbb{R}$. Then (λ, z) is an eigenpair of B if (λ, z') is an eigenpair of B.
   (c) Let $\lambda \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$. Then $(\lambda, u + iv)$ is an eigenpair of A if (λ, z) and (λ, z') are eigenpairs of B.

   (4 pts.)

3. Let $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \in \mathbb{C}^{n \times k}$, $k \leq 2n$ be isometric, $U_1 \in \mathbb{C}^k$. Moreover let U_1 have the singular values $\alpha_1 \geq \ldots \geq \alpha_k$ and U_2 the singular values $\beta_1 \leq \ldots \leq \beta_k$. Then
$$\alpha_i^2 + \beta_i^2 = 1 \text{ for all } i.$$

   (3 pts.)

4. Let $\mathcal{U}, \mathcal{V} \subseteq \mathbb{C}^n$ be k-dimensional subspaces and $x_1, \ldots, x_k$, $y_1, \ldots, y_k$ the corresponding canonical vectors. Show that
$$\langle x_i, y_j \rangle = 0, \text{ for } i \neq j.$$

   (5 pts.)

Programming assignment: PA9 (05-06.07.2011) **(10 pts.)**

1. Using the singular value decomposition find the rank-k approximation of the image matrix associated with the image `gatlin`. Use

   `load  gatlin`

   in Matlab to load the image data. For this image matrix

   (a) show all the singular values (plot $\sigma_1, \sigma_2, \ldots, \sigma_p$, $p = \min\{m, n\}$, use Matlab `semilogy`),

   (b) show images for low-rank approximations with $k = 100, 25, 10, 1$,

   (c) calculate the error $\|A - A_k\|_2$ and compare it to σ_{k+1}.

   Use Matlab commands `image` and `colormap(map)` to display images.

2. Write the program `powermethod.m` which computes the largest eigenvalue and the corresponding eigenvector for a given matrix using the *power method.*

   (a) Test your program for matrix A and starting vector q, i.e.,

   $$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 9 & 2 \\ 0 & -1 & 2 \end{bmatrix}, \quad q = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

   Perform at least 10 iterations.

   (b) Compute the dominant eigenvector v using `[V,D] = eig(A)` and scale it such that you can compare it with q_k. Compare $|\frac{\lambda_2}{\lambda_1}|$ with

   $$\frac{\|q_{k+1} - v\|}{\|q_k - v\|}, \quad k = 1, 2, \ldots.$$

   (c) Repeat (a) and (b) for matrices

   $$B = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 9 & 2 \\ -4 & -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 3 & 2 \\ -4 & -1 & 2 \end{bmatrix}.$$

   What difference do you observe and why? For the matrix B compare $|\frac{\lambda_2}{\lambda_1}|$ with

   $$\sqrt[k]{\frac{\|q_{k+1} - v\|}{\|q_0 - v\|}}, \quad k = 1, 2, \ldots.$$