Technische Universität Berlin **Sommersemester 2011**
Fakultät II – Mathematik und Naturwissenschaften
Institut für Mathematik
Prof. Dr. Christian Mehl, Dr. Agnieszka Miedlar Stand: June 30, 2011

Numerische Mathematik II/ Numerical Analysis II
10. Assignment

Homework: HW10 (07.07.2011)

1. **(Distance between subspaces)**

   Let $\mathcal{S}, \mathcal{T} \subseteq \mathbb{C}^n$ be the nontrivial k-dimensional subspaces and let $V \in \mathbb{C}^{n\times n}$ be nonsingular. Show that
   $$d(V^{-1}\mathcal{S}, V^{-1}\mathcal{T}) \leq \kappa(V) d(\mathcal{S}, \mathcal{T}).$$

   (3 pts.)

2. **(Subspaces and matrix representations)**

   Let
   $$\mathcal{V} = \mathcal{R}\left(\begin{bmatrix} 0 \\ I_{n-m} \end{bmatrix}\right), \quad W = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \quad Z_1 \in \mathbb{C}^{m\times m}, Z_2 \in \mathbb{C}^{(n-m)\times m},$$
   where $\operatorname{rank}(W) = m$. Show that
   $$\mathcal{R}(W) \cap \mathcal{V} = \{0\} \quad \Longleftrightarrow \quad Z_1 \text{ is nonsingular.}$$

   (4 pts.)

3. Let the Rayleigh quotient $r(w)$ of the symmetric matrix $A \in \mathbb{R}^{n\times n}$ and vector $w \in \mathbb{R}^n$ be defined as
   $$r(w) = \frac{w^T A w}{w^T w}.$$
   Show that
   $$\operatorname{grad} r(w) = \frac{2}{w^T w}\Big(Aw - r(w)w\Big).$$
   Moreover, conclude that for $v \in \mathbb{R}^n$ being an eigenvector of A
   $$|r(w) - r(v)| = \mathcal{O}\Big(\|w - v\|^2\Big) \text{ for } w \to v.$$

   (4 pts.)

4. Let
   $$P = I - \beta v v^T, \quad \beta = \frac{2}{v^T v}, \quad v \in \mathbb{R}^n$$
   be the Householder transformation.

   (a) Show that P is symmetric and orthogonal.

   (b) Show that choosing $v = x + \alpha e_1$ leads to
   $$v = x \pm \|x\| e_1 \quad \text{and} \quad Px = \mp \|x\| e_1.$$

(c) Let $C = PBP \in \mathbb{R}^{n\times n}$, with symmetric matrix $B \in \mathbb{R}^{n\times n}$. Show that

$$C = B - vq^T - qv^T,$$

where $p = \beta Bv, \quad q = p - \frac{\beta(p^T v)}{2} v.$

(4 pts.)

Programming assignment: PA10 (12.07.2011) **(10 pts.)**

1. Write the program which finds the eigenvalues of the given real matrix using the QR algorithm
   (a) without shifts,
   (b) with real shifts.

   Test your program for matrices

   (a)
   $$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
   without shift, with shift $\mu = 0.1$, with Wilkinson shift $\mu = \pm 1$. What do you observe and why?

   (b)
   $$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
   without shift, with shift $\mu = 0.1$. What do you observe and why?

2. Write the program which reduces a given real matrix into the Hessenberg form using the Householder transformation. Compare your implementation with Matlab function `hess`. Test your program for

$$A = \begin{bmatrix} -149 & -50 & -154 \\ 537 & 180 & 546 \\ -27 & -9 & -25 \end{bmatrix} \quad \text{and} \quad B = \texttt{magic(7)}.$$