Corrections:

Examples: for Nyström methods: $u_{j+k} = u_{j+r-\ell} + h \sum_{i=0}^{r} \beta_i^{(r,\ell)} f(t_{j+i}, u_{j+i})$

$k=1$: $u_{j+1} = u_{j-1} + 2h f(t_j, u_j)$ midpoint-rule

(exceptional case: 2-step method although $k=1$)

$k=2$: $u_{j+2} = u_j + 2hf(t_{j+1}, u_{j+1})$ midpoint-rule (again)

$k=3$: $u_{j+3} = u_{j+1} + \frac{h}{3}\left(7f(t_{j+2}, u_{j+2}) - 2f(t_{j+1}, u_{j+1}) + f(t_j, u_j)\right)$

Corollary: For a given $\varrho(z) = \sum_{\ell=0}^{k} \alpha_\ell z^\ell$, $\alpha_k \neq 0$ with $\varrho(1) = 0$ there exists a unique polynomial $\sigma(z) = \sum_{\ell=0}^{r} \beta_\ell z^\ell$ such that the corresponding k-step method has order $p = r+1$.

Proof: Let $p = r+1$. We obtain the linear system

$$-\underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2\cdot 2 & \cdots & 2\cdot r \\ \vdots & \vdots & & \vdots \\ p & p\cdot 2^{p-1} & \cdots & p\cdot r^{p-1} \end{bmatrix}}_{=:\, B \in \mathbb{R}^{p,p} = \mathbb{R}^{r+1,r+1}} \cdot \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} = b(\alpha_0, \dots, \alpha_k)$$

Then: $[\gamma_1, \dots, \gamma_p] \cdot B = 0$

$\iff q'(\ell) = 0$ for $\ell = 0, \dots, r$ where $q(z) = \sum_{j=1}^{p} \gamma_j z^j$

$\Rightarrow q'$ has $r+1$ roots

$\Rightarrow q' \equiv 0$ as $\deg(q') = \deg(q) - 1 \leq p-1 = r$

$\Rightarrow q$ is constant

$\Rightarrow \gamma_1 = \dots = \gamma_p = 0$

so B is invertible

Divergence of a multistep method

We consider the explicit linear multistep method:

$$u_{j+2} + 4u_{j+1} - 5u_j = h(4f(t_{j+1}, u_{j+1}) + 2f(t_j, u_j))$$

for the initial value problem $y' = -y,\ y(0) = 1$ which has the exact solution $y(t) = e^{-t}$. As initial values for our method, we use the exact values $u_0 = 1$ and $u_1 = e^{-h}$.

t	exact solution	h=0.1	h=0.01	h=0.005
0	1.000	1.000	1.000	1.000
0.1	0.9048	0.9048	0.9044	-289.9512
0.2	0.8187	0.8187	-4.9039e+03	-2.9452e+16
0.4	0.6703	0.6700	-5.2730e+17	-3.0199e+44
0.6	0.5488	0.5399	-5.6690e+31	-3.0966e+72
0.8	0.4493	0.1990	-6.0947e+45	-3.1751e+100
1.0	0.3679	-6.6773	-6.5524e+59	-3.2557e+128
1.2	0.3012	-197.9577	-7.0444e+73	-3.3383e+156