solutions of a difference equation:

given: $u_{j+2} + (4+4h)u_{j+1} - (5-2h)u_j = 0$, $j \geq 0$, $u_0 = 1, u_1 = e^{-h}$

① roots of $\psi_h(z) = z^2 + (4+4h)z - (5-2h)$ are

$$z_{1/2} = -2-2h \pm \sqrt{4+8h+4h^2+5-2h}$$
$$= -2-2h \pm 3\sqrt{1+\tfrac{2}{3}h+\tfrac{4}{9}h^2}$$

using $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{4}\cdot\frac{x^2}{2} + \frac{3}{8}\cdot\frac{x^3}{6} - \frac{15}{16}\cdot\frac{x^4}{24} + \mathcal{O}(x^5)$ (Taylor)

we obtain:

$$z_1 = -2-2h+3\left(1+\tfrac{1}{2}\left(\tfrac{2}{3}h+\tfrac{4}{9}h^2\right)-\tfrac{1}{8}\left(\tfrac{2}{3}h+\tfrac{4}{9}h^2\right)^2+\tfrac{1}{16}\left(\tfrac{2}{3}h+\tfrac{4}{9}h^2\right)^3-\tfrac{5}{128}\left(\tfrac{2}{3}h+\tfrac{4}{9}h^2\right)^4+\mathcal{O}(h^5)\right)$$
$$= -2-2h+3\left(1+\tfrac{1}{3}h+\tfrac{2}{9}h^2-\tfrac{1}{18}h^2-\tfrac{2}{27}h^3-\tfrac{2}{81}h^4+\tfrac{1}{54}h^3+\tfrac{1}{27}h^4+\mathcal{O}(h^5)-\tfrac{5}{648}h^4+\mathcal{O}(h^5)\right)$$
$$= 1-h+\tfrac{1}{2}h^2-\tfrac{1}{6}h^3+\tfrac{1}{72}h^4+\mathcal{O}(h^5) = \underline{\underline{1-h+\mathcal{O}(h^2)}}$$
$$z_2 = -2-2h-3(\cdots)$$
$$= \underline{\underline{-5-3h+\mathcal{O}(h^2)}}$$

② solution of the difference equation is

$$u_j = c_1 z_1^j + c_2 z_2^j \quad , \quad j \geq 0$$

we can compute c_1, c_2 from the initial values:

$$u_0 = 1 = c_1 + c_2$$
$$u_1 = e^{-h} = c_1 z_1 + c_2 z_2 = z_1 + c_2(z_2 - z_1) = c_1(z_1 - z_2) + z_2$$

$$\Rightarrow c_1 = \frac{z_2 - e^{-h}}{z_2 - z_1} = \frac{-5-3h-(1-h)+\mathcal{O}(h^2)}{-6\cdot\left(1+\frac{1}{3}h+\mathcal{O}(h^2)\right)} \qquad \text{using } e^{-h} = 1-h+\mathcal{O}(h^2)$$

$$= -\frac{1}{6}\cdot\left(-6-2h+\mathcal{O}(h^2)\right)\cdot\left(1-\frac{1}{3}h+\mathcal{O}(h^2)\right) \qquad \text{using } \frac{1}{1+x} = 1-x+\mathcal{O}(x^2)$$

$$= -\frac{1}{6}\cdot\left(-6+2h-2h+\mathcal{O}(h^2)\right) = \underline{\underline{1+\mathcal{O}(h^2)}}$$

and $c_2 = \dfrac{e^{-h} - z_1}{z_2 - z_1}$

$$= \frac{1-h+\frac{1}{2}h^2-\frac{1}{6}h^3+\frac{1}{24}h^4+\mathcal{O}(h^5)-\left(1-h+\frac{1}{2}h^2-\frac{1}{6}h^3+\frac{1}{72}h^4+\mathcal{O}(h^5)\right)}{-6\left(1+\frac{1}{3}h+\mathcal{O}(h^2)\right)}$$

$$= -\frac{1}{6}\left(\frac{1}{36}h^4+\mathcal{O}(h^5)\right)\cdot\left(1-\frac{1}{3}h+\mathcal{O}(h^2)\right) = \underline{\underline{-\frac{1}{216}h^4+\mathcal{O}(h^5)}}$$