Hessenberg reduction

$$A = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}$$

Let a_1 be the first column of A.

Hessenberg reduction

$$PA = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{bmatrix}$$

We can compute a Householder matrix P that maps a_1 to the first unit coordinate vector.

Hessenberg reduction

$$PAP^{-1} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}$$

But if we do a similarity transformation with P then PAP^{-1} is full again.

Hessenberg reduction

$$A = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}$$

Therefore, compute a Householder matrix $\hat{P}_1 \in \mathbb{C}^{(n-1)\times(n-1)}$ that maps the **red** column vector to the first unit coodinate vector and consider

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & \hat{P}_1 \end{bmatrix}.$$

Hessenberg reduction

$$\begin{bmatrix} 1 & 0 \\ 0 & \hat{P}_1 \end{bmatrix} A = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{bmatrix}$$

Multiplication by P_1 only manipulates the **red** part of the matrix.

Hessenberg reduction

$$\begin{bmatrix} 1 & 0 \\ 0 & \hat{P}_1 \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & \hat{P}_1 \end{bmatrix}^{-1} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{bmatrix}$$

Analogously, multiplication by P_1^{-1} from the right only manipulates **this red part** of the matrix.

Hessenberg reduction

$$P_1 A P_1^{-1} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{bmatrix}$$

Now determine a Householder matrix $\hat{P}_2 \in \mathbb{C}^{n-2 \times n-2}$ that maps the lower **red** part of the second column to the first unit coordinate vector and consider

$$P_2 = \begin{bmatrix} I_2 & 0 \\ 0 & \hat{P}_2 \end{bmatrix}.$$

Hessenberg reduction

$$P_2 P_1 A P_1^{-1} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \end{bmatrix}$$

Multiplication by P_2 only manipulates the **red** part of the matrix.

Hessenberg reduction

$$P_2P_1AP_1^{-1}P_2^{-1} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \end{bmatrix}$$

Analogously, multiplication by P_2^{-1} from the right only manipulates **this red part** of the matrix.

Hessenberg reduction

$$P_2 P_1 A P_1^{-1} P_2^{-1} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \end{bmatrix}$$

and so on...

$$P_3 = \begin{bmatrix} I_3 & 0 \\ 0 & \hat{P}_3 \end{bmatrix}.$$

Hessenberg reduction

$$P_3 P_2 P_1 A P_1^{-1} P_2^{-1} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \end{bmatrix}$$

and so on...

Hessenberg reduction

$$P_3 P_2 P_1 A P_1^{-1} P_2^{-1} P_3^{-1} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \end{bmatrix}$$

and so on...

Hessenberg reduction

$$P_3 \cdots P_1 A P_1^{-1} \cdots P_3^{-1} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \end{bmatrix}$$

and so on...

$$P_4 = \begin{bmatrix} I_4 & 0 \\ 0 & \hat{P}_4 \end{bmatrix}.$$

Hessenberg reduction

$$P_4 \cdots P_1 A P_1^{-1} \cdots P_3^{-1} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \end{bmatrix}$$

and so on...

Hessenberg reduction

$$P_4 \cdots P_1 A P_1^{-1} \cdots P_4^{-1} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \end{bmatrix}$$

READY!

Hessenberg reduction

$$P_4 \cdots P_1 A P_1^{-1} \cdots P_4^{-1} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \end{bmatrix}$$

Summary: The algorithms computes a Hessenberg matrix that is similar to A in finitely many steps.

cost: ca. $\frac{10}{3}n^3$ flops

if also $Q = P_{n-2} \cdots P_1$ is needed explicitly: additional $\frac{4}{3}n^3$ flops