QR iteration on Hessenberg matrices

$$A = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \end{bmatrix}$$

Let an $n \times n$ Hessenberg matrix A be given.

Then with the help of Givens rotations, we can do a QR iteration in only $\mathcal{O}(n^2)$ flops.

QR iteration on Hessenberg matrices

$$A = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \end{bmatrix}$$

Consider the **red** 2×2 submatrix and compute a Givens rotation $\hat{G}_1$ that eliminates the $(2,1)$-element:

$$\hat{G}_1 \begin{bmatrix} * & * \\ * & * \end{bmatrix} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

QR iteration on Hessenberg matrices

$$G_1 A = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \end{bmatrix}$$

Embed $\hat{G}_1$ into an $n \times n$ matrix:

$$G_1 = \begin{bmatrix} \hat{G}_1 & 0 \\ 0 & I_{n-2} \end{bmatrix}$$

Multiplication by G_1 only manipulates the **red** part of the matrix.

QR iteration on Hessenberg matrices

$$G_1 A = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \end{bmatrix}$$

and so on...

$$G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \hat{G}_2 & 0 \\ 0 & 0 & I_{n-3} \end{bmatrix},$$

where $\hat{G}_2$ is a Givens rotation that eliminates the $(2,1)$-entry of the **red** matrix.

QR iteration on Hessenberg matrices

$$G_2 G_1 A = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \end{bmatrix}$$

and so on...

QR iteration on Hessenberg matrices

$$G_3G_2G_1A = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \end{bmatrix}$$

and so on...

QR iteration on Hessenberg matrices

$$G_4G_3G_2G_1A = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{bmatrix}$$

and so on...

QR iteration on Hessenberg matrices

$$G_5 G_4 G_3 G_2 G_1 A = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

and so on...

QR iteration on Hessenberg matrices

$$\underbrace{G_5 G_4 G_3 G_2 G_1}_{Q^*} A = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

Thus, we computed a QR decomposition of A with

$$Q = G_1^{-1} \cdots G_{n-1}^{-1}, \quad R = Q^* A.$$

Now we compute the product RQ, i.e., we form the product

$$RG_1^{-1} \cdots G_{n-1}^{-1} = RG_1^* \cdots G_{n-1}^*$$

QR iteration on Hessenberg matrices

$$R = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

Multiplication by

$$G_1^* = \begin{bmatrix} \hat{G}_1^{\;*} & 0 \\ 0 & I_{n-2} \end{bmatrix}$$

from the right only manipulates the **red** part of the matrix.

QR iteration on Hessenberg matrices

$$RG_1^* = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

QR iteration on Hessenberg matrices

$$RG_1^* = \begin{bmatrix} * & {\color{red}*} & {\color{red}*} & * & * & * \\ * & {\color{red}*} & {\color{red}*} & * & * & * \\ 0 & {\color{red}0} & {\color{red}*} & * & * & * \\ 0 & {\color{red}0} & {\color{red}0} & * & * & * \\ 0 & {\color{red}0} & {\color{red}0} & 0 & * & * \\ 0 & {\color{red}0} & {\color{red}0} & 0 & 0 & * \end{bmatrix}$$

and so on...

Multiplication by G_2^* from the right only manipulates **this red part** of the matrix.

QR iteration on Hessenberg matrices

$$RG_1^*G_2^* = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

and so on...

QR iteration on Hessenberg matrices

$$RG_1^*G_2^*G_3^* = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

and so on...

QR iteration on Hessenberg matrices

$$RG_1^*G_2^*G_3^*G_4^* = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

and so on...

QR iteration on Hessenberg matrices

$$RG_1^*G_2^*G_3^*G_4^*G_5^* = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \end{bmatrix}$$

READY!

Summary: A QR iteration on a Hessenberg matrix A costs only $\mathcal{O}(n^2)$ flops and the resulting matrix is again a Hessenberg matrix.