Householder QR decomposition

$$A = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}$$

Compute a Householder matrix P_1 that maps the first column of A to the first unit coordinate vector.

Householder QR decomposition

$$P_1 A = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{bmatrix}$$

Householder QR decomposition

$$P_1 A = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{bmatrix}$$

Now consider the lower left $(n-1) \times (n-1)$ submatrix. Compute a Householder matrix $\hat{P}_2 \in \mathbb{C}^{(n-1)\times(n-1)}$ that maps the first column of the red submatrix to the first unit coordinate vector.

Householder QR decomposition

$$P_1 A = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{bmatrix}$$

Embed $\hat{P}_2$ into an $n \times n$ matrix:

$$P_2 = \begin{bmatrix} 1 & 0 \\ 0 & \hat{P}_2 \end{bmatrix}$$

Multiplication with P_2 only manipulates the red part of the matrix.

Householder QR decomposition

$$P_2 P_1 A = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \end{bmatrix}$$

Householder QR decomposition

$$P_2 P_1 A = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \end{bmatrix}$$

And so on: Consider again a submatrix (**red**) and compute an adequate Householder matrix

$$P_3 = \begin{bmatrix} I_2 & 0 \\ 0 & \hat{P}_3 \end{bmatrix}$$

Householder QR decomposition

$$P_3 P_2 P_1 A = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \end{bmatrix}$$

And so on...

Householder QR decomposition

$$P_3 P_2 P_1 A = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \end{bmatrix}$$

And so on...

Householder QR decomposition

$$P_4 P_3 P_2 P_1 A = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{bmatrix}$$

And so on...

Householder QR decomposition

$$P_4 P_3 P_2 P_1 A = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & \color{red}{*} & \color{red}{*} \\ 0 & 0 & 0 & 0 & \color{red}{*} & \color{red}{*} \end{bmatrix}$$

And so on...

Householder QR decomposition

$$P_5 P_4 P_3 P_2 P_1 A = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

And so on... READY!

Householder QR decomposition

$$P_5 P_4 P_3 P_2 P_1 A = \begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

Summary: The algorithm computes a QR decomposition of the given $n \times n$ matrix A in finitely many steps.

cost: ca. $\frac{4}{3}n^3$ flops

if $Q = P_{n-1} \cdots P_1$ has to be computed explicitly: additional $\frac{4}{3}n^3$ flops