Technische Universität Berlin **Sommersemester 2012**
Fakultät II – Mathematik und Naturwissenschaften
Institut für Mathematik
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Systems and control theory

Series 1

Task 1: ⊕

1. Show that every behavior given in kernel or image representation defines a linear time-invariant dynamical system.

2. Let $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{\ell,m}$, and $D \in \mathbb{R}^{\ell,m}$. Give a kernel representation of the state-space system
$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$
i.e., the system consisting of all trajectories $(y, x, u) \in \mathcal{C}_\infty^{\ell+n+m}$ such that (1) holds.

3. Give a kernel representation (in the style of 2.) of the system $M\ddot{x} + D\dot{x} + Kx = Bu$, where $M, D, K \in \mathbb{R}^{n,n}$ and $B \in \mathbb{R}^{n,m}$.

Task 2: ⊕
Show that the product of two unimodular matrices is again unimodular.

Task 3: ⊖
We call a matrix *elementary unimodular* if it is a square matrix that (when multiplied from the right) does one of the following:

a) exchanges two rows;

b) multiplies a row by a non-zeros constant;

c) adds to one row i the a-multiple of *another* row $j \neq i$, where $a \in \mathbb{C}[\lambda]$ is arbitrary.

Understand, why elementary unimodular matrices are indeed unimodular. Then, use the techniques from the proof of the Smith canonical form to show the following:

1. For every polynomial matrix $P \in \mathbb{C}[\lambda]^{p,q}$ there exists a unimodular matrix $U \in \mathbb{C}[\lambda]^{p,p}$ (which can be written as the finite product of elementary unimodular matrices) such that UP is upper triangular.

2. Use 2. to prove that every unimodular matrix can be written as the product of a finite number of elementary unimodular matrices.

Task 4: ⊕
Let $P \in \mathbb{C}[\lambda]^{p,q}$. Show that for $\bullet \in \{\infty, c\}$ the following holds:

1. If $z \in \mathcal{C}_\bullet^q$ then also $P\left(\frac{d}{dt}\right) z \in \mathcal{C}_\bullet^p$.

2. If $z \in \mathcal{C}_\bullet^q$ then $P\left(\frac{d}{dt}\right) Q\left(\frac{d}{dt}\right) z = (PQ)\left(\frac{d}{dt}\right) z$.

3. If $z \in \mathcal{C}_\bullet^q$ and $U \in \mathbb{C}[\lambda]^{q,q}$ is unimodular, then $U^{-1}\left(\frac{d}{dt}\right)\left(U\left(\frac{d}{dt}\right) z\right) = z$.

Remark: Some professors claim that 2. is trivial.

Task 5: ⊕

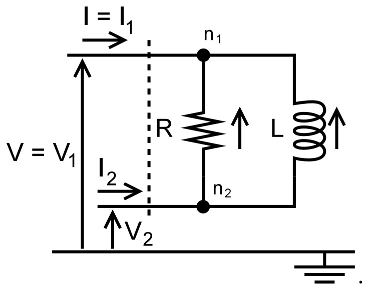
1. Let $P \in \mathbb{C}[\lambda]^{p,q}$ and let $S \in \mathbb{C}[\lambda]^{p,p}$ and $T \in \mathbb{C}[\lambda]^{q,q}$ both be unimodular. Show that
$$\mathcal{B}(SPT) = \mathcal{B}(PT) = T^{-1}\left(\tfrac{d}{dt}\right)\mathcal{B}(P).$$
and conclude that $\mathcal{B}(P) = T\left(\tfrac{d}{dt}\right)\mathcal{B}(PT)$.
2. Let $U \in \mathbb{C}[\lambda]^{q,m}$ and let $T \in \mathbb{C}[\lambda]^{q,q}$ and $S \in \mathbb{C}[\lambda]^{m,m}$ both be unimodular. Show that
$$\text{image}_{\mathcal{C}_\infty}\left(T^{-1}US\right) = \text{image}_{\mathcal{C}_\infty}\left(T^{-1}U\right) = T^{-1}\text{image}_{\mathcal{C}_\infty}(U).$$

Task 6: ⊚

Let $\mathcal{A} \subset \mathcal{C}_\infty^r$ and $\mathcal{B} \subset \mathcal{C}_\infty^{q-r}$ be linear subspaces. Let $T_1 \in \mathbb{C}[\lambda]^{q,r}$ and $T_2 \in \mathbb{C}[\lambda]^{q,q-r}$ be such that $\begin{bmatrix} T_1 & T_2 \end{bmatrix}$ is unimodular. Show that
$$\begin{bmatrix} T_1 & T_2 \end{bmatrix}\left(\frac{d}{dt}\right)\begin{bmatrix} \mathcal{A} \\ \mathcal{B} \end{bmatrix} = \left(T_1\left(\frac{d}{dt}\right)\mathcal{A}\right) \oplus \left(T_2\left(\frac{d}{dt}\right)\mathcal{B}\right),$$
where $\begin{bmatrix} \mathcal{A} \\ \mathcal{B} \end{bmatrix} := \mathcal{A} \times \mathcal{B}$ denotes the Cartesian product.

Task 7: ⊕

Use the methodology introduced in the "Motivation"-slides to deduce the behavioral equations for the circuit



Show that $I_1 = -I_2$ and eliminate I_2 from the system. Then fix the lower wire to the ground, i.e., impose $V_2 = 0$, and eliminate V_2 and n_2 form the system, so that with the definitions
$$P(\lambda) := \begin{bmatrix} 1 & 1 & 1 & 0 \\ -R & 0 & 0 & 1 \\ 0 & \lambda L & 0 & -1 \end{bmatrix} \text{ and } z := \begin{bmatrix} I_R \\ I_L \\ I \\ V \end{bmatrix} \tag{2}$$
the system is described by $P(\frac{d}{dt})z(t) = 0$.

1. Compute the Smith canonical form of P as $S(\lambda)P(\lambda)T(\lambda) = D(\lambda)$. Show that the transformation matrices S and T are indeed unimodular.
2. Compute a kernel-spanning matrix $U \in \mathbb{C}[\lambda]^{4,1}$ for P.
3. Give an image representation of the system.
4. Compute the echelon form of P.

Task 8: ⊚

Compute the echelon form of
$$R(\lambda) := \begin{bmatrix} 0 & 0 \\ 1 & \frac{z-1}{z} \\ \frac{1}{z} & \frac{z-1}{z^2} \end{bmatrix} \in \mathbb{C}(\lambda)^{3,2}$$